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Unique solvability of the initial boundary value problems for compressible viscous fluids

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Abstract

We study the Navier–Stokes equations for compressible barotropic fluids in a domain $\Omega \subset \mathbb{R}^3$. We first prove the local existence of the unique strong solution, provided the initial data satisfy a natural compatibility condition. The initial density needs not be bounded away from zero; it may vanish in an open subset (*vacuum*) of Ω or decay at infinity when Ω is unbounded. We also prove a blow-up criterion for the local strong solution, which is new even for the case of positive initial densities. Finally, we prove that if the initial vacuum is not so irregular, then the compatibility condition of the initial data is necessary and sufficient to guarantee the existence of a unique strong solution.

Résumé

Nous étudions des problèmes de Navier–Stokes dans un ouvert $\Omega \subset \mathbb{R}^3$ pour des fluides compressibles, barotropiques. Nous démontrons, tout d'abord, l'existence et l'unicité de la solution forte locale si les conditions initiales vérifient une condition de compatibilité naturelle. La densité initiale n'est pas nécessairement bornée loin de zéro, elle pourrait ou bien être nulle dans un ouvert, ce qui correspond à un vide de Ω, ou bien s'évanouir à l'infini si Ω n'est pas borné. Nous établissons aussi un critère d'explosion de la solution forte locale, ce résultat est nouveau, même pour des densités initiales positives. Enfin, nous montrons que si le sous ensemble ouvert vide initial n'est pas top irrégulier, alors la condition de compatibilité sur les données initiales est nécessaire et suffisante pour l'existence de la solution forte qui est alors unique.

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1. Introduction

The motion of a compressible viscous barotropic fluid in a domain Ω of $\mathbb{R}^3$ can be described by the system of equations, known as the *Navier–Stokes equations*:

$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = 0, & \\ (\rho u)_t + \operatorname{div}(\rho u \otimes u) + Lu + \nabla p = \rho f & \text{in } (0,T)\times\Omega, \\ Lu = -\mu\Delta u - (\lambda+\mu)\nabla \operatorname{div} u, & p = p(\rho), \end{cases} \tag{1.1}$$

and the initial and boundary conditions:

$$\begin{cases} (\rho,u)|_{t=0} = (\rho_0,u_0) & \text{in } \Omega;\ u = 0 \text{ on } (0,T)\times\partial\Omega, \\ \rho(t,x)\to 0, \quad u(t,x)\to 0 & \text{as } |x|\to\infty,\ (t,x)\in(0,T)\times\Omega. \end{cases} \tag{1.2}$$

Here ρ, u and p denote the unknown density, velocity and pressure, respectively. The motion of the fluid is driven by an external force f and is characterized by the viscosity coefficients μ and λ. We assume that μ and λ are fixed constants satisfying the physical restrictions

$$\mu > 0 \quad \text{and} \quad 3\lambda + 2\mu \geqslant 0.$$

In this paper, we study strong solutions to the initial boundary value problem (1.1), (1.2) with nonnegative initial densities. In case that the data ρ_0, u_0, f are sufficiently regular and the initial density ρ_0 has a positive lower bound, there exists a unique *local* strong solution to the problem (1.1), (1.2) and the solution exists globally in time provided that the data are small in some sense. For details, we refer the readers to the papers [4,9,10,13–17,19, 21,23,24].

On the other hand, there have been few existence results on the strong solutions for the general case of nonnegative initial densities. The first result was proved by R. Salvi and I. Straškraba. They showed in [17] that if Ω is a bounded domain, $p = p(\cdot) \in C^2[0,\infty)$, $\rho_0 \in H^2$, $u_0 \in H_0^1 \cap H^2$ and the compatibility condition:

$$Lu_0 + \nabla p(\rho_0) = \rho_0^{1/2} g, \quad \text{for some } g \in L^2, \tag{1.3}$$

is satisfied, then there exists a unique local strong solution (ρ,u) to the initial boundary value problem (1.1), (1.2). Independently of their work, H.J. Choe and H. Kim [3] proved a similar existence result when Ω is either a bounded domain or the whole space, $p = a\rho^\gamma$ $(a>0,\ \gamma>1)$, $\rho_0 \in L^1 \cap H^1 \cap W^{1,6}$, $u_0 \in D_0^1 \cap D^2$ and the condition (1.3) is satisfied.

Throughout this paper, we use the following simplified notations for standard homogeneous and inhomogeneous Sobolev spaces:

$$L^r = L^r(\Omega), \qquad W^{k,r} = W^{k,r}(\Omega), \qquad H^k = W^{k,2},$$
$$D^{k,r} = \{v \in L^1_{\mathrm{loc}}(\Omega): |\nabla^k v|_{L^r} < \infty\}, \qquad D^k = D^{k,2},$$
$$D^1_0 = \{v \in L^6: |\nabla v|_{L^2} < \infty \text{ and } v = 0 \text{ on } \partial\Omega\}, \qquad H^1_0 = D^1_0 \cap L^2.$$

Note that if Ω is a bounded domain of $\mathbb{R}^3$ with smooth boundary, then $D^1_0 = H^1_0$ and $D^{k,r} = W^{k,r}$. For a proof, see G. Galdi [8].

The purpose of the paper is to provide a *local theory* of strong solutions with nonnegative densities in the framework of Sobolev spaces.

First, we prove the existence and uniqueness of the local strong solutions for a larger class of domains, initial densities and equations of states; Ω is either a bounded domain in $\mathbb{R}^3$ with smooth boundary or an usual unbounded domain such as the whole space $\mathbb{R}^3$, the half space $\mathbb{R}^3_+$ and an exterior domain with smooth boundary, $\rho_0 \in H^1 \cap W^{1,q}$ for some q with $3 < q < \infty$, and $p = p(\cdot) \in C^1[0,\infty)$. To prove both existence and uniqueness of a strong solution of the compressible Navier–Stokes equations (1.1), it is essential to show that the density is bounded above. For the density may blow up in finite time (see [25,26]) and the most general uniqueness results require the boundedness of the density (for instance, see [3,5]). Moreover, since the Sobolev embedding $W^{1,q} \hookrightarrow L^\infty$ holds only for $q > 3$, the $W^{1,q}$-regularity of the initial density seems inevitable to obtain a strong solution in the framework of Sobolev spaces. The additional L^1-integrability in [3] is removed by means of suitable cut-off functions. But the H^1-regularity is necessary to prove one of key estimates for the existence and uniqueness in case of unbounded domains. See the derivation of (3.27).

The second main result is a blow-up criterion of the local strong solution. We prove that if T^* is the maximal existence time of the local strong solution (ρ, u) and $T^* < T$, then

$$\limsup_{t \nearrow T^*}\left(|\rho(t)|_{H^1 \cap W^{1,q_0}} + |u(t)|_{D^1_0}\right) = \infty, \tag{1.4}$$

where $q_0 = \min(6, q)$. B. Desjardins [5] proved the local existence of a weak solution (ρ, u) with a bounded nonnegative density to the periodic boundary value problem for (1.1) as long as $\sup_{0 \leqslant t \leqslant T^*}(|\rho(t)|_{L^\infty(\mathbb{T}^3)} + |\nabla u(t)|_{L^2(\mathbb{T}^3)}) < \infty$. But concerning strong solutions, the blow-up criterion (1.4) is new even for the case of positive initial densities. Our proof of (1.4) is based on the ideas of Y. Cho and H. Kim [2], who studied the incompressible Navier–Stokes equations with a density-dependent viscosity in a bounded domain. Using a classical iteration method, they proved a local existence result on strong solutions and a blow-up criterion analogous to (1.4).

The final result is concerned with the initial condition (1.2) and the compatibility condition (1.3). The local strong solution (ρ, u), existence of which is guaranteed under the condition (1.3), is continuous in a strong topology:

$$\rho \in C([0, T_*]; H^1 \cap W^{1,q_0}) \quad \text{and} \quad u \in C([0, T_*]; D^1_0 \cap D^2).$$

Hence it may be expected that

$$\rho(0) = \rho_0 \quad \text{and} \quad u(0) = u_0, \tag{1.5}$$

where $\rho(0)$ and $u(0)$ are the strong limits of $\rho(t)$ and $u(t)$, respectively, as $t \to 0$. The first identity follows easily since it can be deduced from the weak formulation of the continuity equation, the first equation in (1.1), that $\rho(t)$ converges in a weak sense to ρ_0 as $t \to 0$. See [7,11,12,18]. From the momentum equation, we also deduce that $\rho u(t) \rightharpoonup \rho_0 u_0$ as $t \to 0$ and thus $\rho(0)u(0) = \rho_0 u_0$. However the second identity in (1.5) can not follow from this observation owing to the possible vanishing of the initial density. We show that the compatibility condition (1.3) forces the identity (1.5) to hold for a large class of initial densities. Included are, in particular, the initial densities which are positive in Ω but decay as $x \to \partial\Omega$ or $|x| \to \infty$. Moreover, we show that the condition (1.3) is also necessary to guarantee the existence of a unique strong solution.

The paper is organized as follows. In Section 2, we prove the global existence and regularity of the unique strong solution to a linearized problem of the nonlinear problem (1.1), (1.2). The results are used in Section 3 to construct approximate solutions to the original nonlinear problem. We then derive some uniform bounds in higher norms, prove the convergence and thus obtain a local existence result on strong solutions with positive densities. We also prove that the uniform bounds are independent of lower bounds of the initial density. In Section 4, we state and prove all of our main results—existence, regularity, uniqueness, a blow-up criterion, sense of the initial condition and necessity of the compatibility condition (1.3). The final section, Section 5, is devoted to deriving some regularity estimates for the Lamé operator L, which play crucial roles in proving all the results in case of unbounded domains.

Notation: We denote by C a generic positive constant depending only on q, $|\rho_0|_{H^1\cap W^{1,q}}$, $|u_0|_{D_0^1}$, $|p(\cdot)|_{C^1[0,\infty)}$, $|f|_{C([0,T];L^2)\cap L^2(0,T;L^q)}$, $|f_t|_{L^2(0,T;H^{-1})}$ and T, but independent of lower bounds of the initial density and the size of the domain. Since we consider only local results, we may assume that $T < \infty$.

2. Global existence for the linearized equations

We consider the following linearized system:

$$\rho_t + \operatorname{div}(\rho v) = 0 \quad \text{in } (0,T) \times \Omega, \tag{2.1}$$

$$(\rho u)_t + \operatorname{div}(\rho v \otimes u) + Lu + \nabla p = \rho f \quad \text{in } (0,T) \times \Omega, \tag{2.2}$$

where $Lu = -\mu\Delta u - (\lambda + \mu)\nabla \operatorname{div} u$, $p = p(\rho)$ and v is a known vector field. If the initial density ρ_0 is bounded away from zero, then we can apply standard arguments to prove the global existence of a unique strong solution to the initial boundary value problem (2.1), (2.2) and (1.2), since the system can be uncoupled into a linear transport equation and a linear parabolic system.

In this section, we prove the following existence result for the general case of nonnegative initial densities.

Theorem 1. *Assume that $p = p(\cdot) \in C^1[0,\infty)$, and the data (ρ_0, u_0, f) satisfy the regularity conditions*:

$$
\begin{aligned}
&0 \leqslant \rho_0 \in H^1 \cap W^{1,q}, \qquad u_0 \in D_0^1 \cap D^2,\\
&f \in C\big(0,T;L^2\big) \cap L^2\big(0,T;L^q\big) \quad \textit{and} \quad f_t \in L^2\big(0,T;H^{-1}\big),
\end{aligned}
\tag{2.3}
$$

for some q *with* $3 < q < \infty$ *and the compatibility condition*:

$$
Lu_0 + \nabla p(\rho_0) = \rho_0^{1/2} g \quad \textit{for some } g \in L^2. \tag{2.4}
$$

If in addition, v *satisfies the regularity conditions*:

$$
v \in L^\infty\big(0,T;D_0^1 \cap D^2\big) \cap L^2\big(0,T;D^{2,q_0}\big) \quad \textit{and} \quad v_t \in L^2\big(0,T;D_0^1\big),
$$

where $q_0 = \min(6,q)$, *then there exists a unique strong solution* (ρ,u) *to the initial boundary value problem* (2.1), (2.2), (1.2) *such that*

$$
\begin{aligned}
&\rho \in C\big([0,T];H^1 \cap W^{1,q_0}\big), \qquad \rho_t \in L^\infty\big(0,T;L^2 \cap L^{q_0}\big),\\
&u \in C\big([0,T];D_0^1 \cap D^2\big) \cap L^2\big(0,T;D^{2,q_0}\big),\\
&u_t \in L^2\big(0,T;D_0^1\big) \quad \textit{and} \quad \sqrt{\rho}\, u_t \in L^\infty\big(0,T;L^2\big).
\end{aligned}
\tag{2.5}
$$

Remark 2. If v satisfies the additional regularity $v \in C([0,T];D_0^1 \cap D^2)$, then we can also show that $\rho_t \in C([0,T];L^2 \cap L^{q_0})$.

We first prove the existence of the strong solutions for bounded domains. Then the case of unbounded domains can be proved by means of the standard domain expansion technique. Finally, we prove the uniqueness and continuity of the strong solutions.

2.1. Existence for bounded domains

We begin with an existence result for the case of positive initial densities.

Lemma 3. *Let* Ω *be a bounded domain in* $\mathbb{R}^3$ *with smooth boundary, and let* (ρ_0,u_0,f) *be a given data satisfying the regularity condition* (2.3). *Assume further that* $v \in L^\infty(0,T;D_0^1 \cap D^2) \cap L^2(0,T;D^3)$, $v_t \in L^2(0,T;D_0^1)$, $\rho_0 \in H^2$, $p = p(\cdot) \in C^2[0,\infty)$ *and* $\rho_0 \geqslant \delta$ *in* Ω *for some constant* $\delta > 0$. *Then there exists a unique strong solution* (ρ,u) *to the initial boundary value problem* (2.1), (2.2) *and* (1.2) *such that*

$$
\begin{aligned}
&\rho \in C\big([0,T];H^2\big), \quad u \in C\big([0,T];D_0^1 \cap D^2\big) \cap L^2\big(0,T;D^3\big),\\
&\rho_t \in C\big([0,T];H^1\big), \quad u_t \in L^2\big(0,T;D_0^1\big) \cap C\big([0,T];L^2\big)\\
&\textit{and } \rho > 0 \textit{ on } [0,T] \times \overline{\Omega}.
\end{aligned}
\tag{2.6}
$$

Proof. It follows, from a classical embedding result, that $v \in C([0,T];H^2)$. Hence the existence and regularity of a unique solution of the linearized continuity equation (2.1) have been well-known. Moreover, the unique solution ρ can be expressed by:

$$\rho(t,x)=\rho_0\big(U(0,t,x)\big)\exp\left[-\int_0^t \operatorname{div} v\big(s,U(s,t,x)\big)\,\mathrm{d}s\right], \tag{2.7}$$

where $U=U(t,s,x)$ is the solution to

$$\begin{cases} \dfrac{\partial}{\partial t}U(t,s,x)=v\big(t,U(t,s,x)\big), & 0\leqslant t\leqslant T,\\ U(s,s,x)=x, & 0\leqslant s\leqslant T,\ x\in\overline{\Omega}. \end{cases}$$

For a detailed proof, see the papers [23] and [24] by A. Valli. As a consequence of (2.7) and Sobolev inequality, we have:

$$\rho(t,x)\geqslant\delta\exp\left[-\int_0^T \big|\nabla v(s)\big|_{L^\infty}\,\mathrm{d}s\right]>0 \tag{2.8}$$

for $(t,x)\in[0,T]\times\overline{\Omega}$. Hence the linearized momentum equation (2.2) can be written as a linear parabolic system $u_t+v\cdot\nabla u+\rho^{-1}Lu=F$ with the force term $F=f-\rho^{-1}\nabla p$ satisfying $F\in L^2(0,T;H^1)$ and $F_t\in L^2(0,T;H^{-1})$. The existence and regularity of the unique solution u can be proved by applying classical methods, for instance, the method of continuity (see [24]). □

Now we prove the existence of the strong solutions for bounded domains; let Ω be a bounded domain of $\mathbb{R}^3$ with smooth boundary. Without loss of generality, we may assume that $p(0)=0$. Furthermore, assume for the moment that $p=p(\cdot)\in C^2[0,\infty)$, $v\in L^2(0,T;D^3)$, $\rho_0\in H^2$ and $\rho_0\geqslant\delta$ in Ω for some constant $\delta>0$. Then thanks to the previous lemma, there exists a unique strong solution (ρ,u) satisfying the regularity (2.6). To remove the additional hypotheses, we will derive some uniform estimates independent of δ, $|v|_{L^2(0,T;D^3)}$, $|\rho_0|_{D^2}$, $\big|\frac{\mathrm{d}^2p}{\mathrm{d}\rho^2}(\cdot)\big|_{C^0[0,\infty)}$ and the size of the domain Ω.

First, we consider the solution ρ of the linearized continuity equation (2.1). Multiplying (2.1) by ρ^{r-1} ($r=2$ or q_0) and integrating (by parts) over Ω, we obtain:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int|\rho|^r\,\mathrm{d}x\leqslant C\int|\nabla v||\rho|^r\,\mathrm{d}x.$$

Sobolev inequality thus yields:

$$\frac{\mathrm{d}}{\mathrm{d}t}|\rho|_{L^r}^r\leqslant C|\nabla v|_{W^{1,q_0}}|\rho|_{L^r}^r. \tag{2.9}$$

Then differentiating (2.1) with respect to x_j, multiplying by $\partial_j\rho|\partial_j\rho|^{r-2}$, $1\leqslant j\leqslant 3$, and then integrating over Ω, we have:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int|\partial_j\rho|^r\,\mathrm{d}x\leqslant C\int|\nabla v||\nabla\rho|^r+|\rho||\nabla\rho|^{r-1}\big|\nabla^2 v\big|\,\mathrm{d}x.$$

Sobolev inequality also yields:

$$\frac{\mathrm{d}}{\mathrm{d}t}|\partial_j\rho|_{L^r}^r \leqslant C|\nabla v|_{H^1\cap D^{1,q_0}}|\rho|_{H^1\cap W^{1,q_0}}^r, \tag{2.10}$$

where we used the obvious notation

$$|\cdot|_{X\cap Y} = |\cdot|_X + |\cdot|_Y \quad \text{for Banach spaces } X, Y.$$

Using (2.9) and (2.10) together with Gronwall's inequality, we get:

$$\sup_{0\leqslant t\leqslant T}\big|\rho(t)\big|_{H^1\cap W^{1,q_0}} \leqslant |\rho_0|_{H^1\cap W^{1,q_0}} \exp\left(C\int_0^T \big|\nabla v(s)\big|_{H^1\cap D^{1,q_0}}\,\mathrm{d}s\right) \leqslant \widetilde{C}. \tag{2.11}$$

Since $\rho_t = -v\cdot\nabla\rho - \rho\operatorname{div} v$, $p = p(\rho)$ and $p(0) = 0$, we also have:

$$\sup_{0\leqslant t\leqslant T}\left(\big|\rho_t(t)\big|_{L^2\cap L^{q_0}} + \big|p(t)\big|_{H^1\cap W^{1,q_0}} + \big|p_t(t)\big|_{L^2\cap L^{q_0}}\right) \leqslant \widetilde{C}. \tag{2.12}$$

Throughout the proof, we denote by $\widetilde{C}$ a generic positive constant depending only on the norms of v specified in Theorem 1 and parameters of C, but independent of δ, $|v|_{L^2(0,T;D^3)}$, $|\rho_0|_{D^2}$, $\big|\frac{\mathrm{d}^2p}{\mathrm{d}\rho^2}(\cdot)\big|_{C^0[0,\infty)}$ and the size of Ω.

Next, we consider the solution u of the linearized momentum equation (2.2). In view of Eq. (2.1), (2.2) can be rewritten as

$$\rho u_t + \rho v\cdot\nabla u + Lu + \nabla p = \rho f. \tag{2.13}$$

Multiplying this equation by u_t and integrating over Ω, we have:

$$\begin{aligned}&\int \rho|u_t|^2\,\mathrm{d}x + \frac{\mathrm{d}}{\mathrm{d}t}\int \frac{\mu}{2}|\nabla u|^2 + \frac{\lambda+\mu}{2}(\operatorname{div} u)^2 - p\operatorname{div} u\,\mathrm{d}x\\ &\quad = \int (\rho f - \rho v\cdot\nabla u)\cdot u_t - p_t\operatorname{div} u\,\mathrm{d}x.\end{aligned} \tag{2.14}$$

Then, using Young's inequality, we obtain:

$$\begin{aligned}&\frac{1}{2}\int \rho|u_t|^2\,\mathrm{d}x + \frac{\mathrm{d}}{\mathrm{d}t}\int \frac{\mu}{2}|\nabla u|^2 + \frac{\lambda+\mu}{2}(\operatorname{div} u)^2 - p\operatorname{div} u\,\mathrm{d}x\\ &\quad \leqslant \int \rho|f|^2 + \rho|v|^2|\nabla u|^2 + |p_t||\nabla u|\,\mathrm{d}x.\end{aligned} \tag{2.15}$$

By virtue of the estimates (2.11), (2.12) and Sobolev inequality, the right-hand side of (2.15) is bounded above by $\widetilde{C}(1+|\nabla u|_{L^2}^2)$. Hence integrating (2.15) over $(0,t)$, we deduce that

$$\int_0^t \left|\sqrt{\rho}\,u_t\right|_{L^2}^2 \mathrm{d}s + \left|\nabla u(t)\right|_{L^2}^2 \leqslant \widetilde{C} + \widetilde{C}\int_0^t |\nabla u|_{L^2}^2 \,\mathrm{d}s.$$

Therefore, in view of Gronwall's inequality, we have:

$$\int_0^T \left|\sqrt{\rho}\,u_t\right|_{L^2}^2 \mathrm{d}s + \sup_{0\leqslant t\leqslant T}\left|u(t)\right|_{D_0^1}^2 \leqslant \widetilde{C}. \tag{2.16}$$

To derive higher regularity estimates, we differentiate (2.13) with respect to t and obtain:

$$\rho u_{tt} + \rho v\cdot\nabla u_t + Lu_t + \nabla p_t = \rho f_t + \rho_t(f - u_t - v\cdot\nabla u) - \rho v_t\cdot\nabla u.$$

Multiplying this equation by u_t, integrating over Ω and using the linearized continuity equation (2.1), we obtain:

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int \rho|u_t|^2\,\mathrm{d}x + \int \mu|\nabla u_t|^2 + (\lambda+\mu)(\operatorname{div} u_t)^2\,\mathrm{d}x\\
&\quad = \int \big(\rho f_t + \rho_t(f - u_t - v\cdot\nabla u) - \rho v_t\cdot\nabla u\big)\cdot u_t\,\mathrm{d}x + \int p_t \operatorname{div} u_t\,\mathrm{d}x.
\end{aligned}\tag{2.17}$$

For a rigorous derivation of (2.17), see Remark 4 below. Using (2.1) again, we then deduce from (2.17) that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int \rho|u_t|^2\,\mathrm{d}x + C^{-1}\int |\nabla u_t|^2\,\mathrm{d}x\\
&\quad\leqslant C\int \big[\rho\big(|f_t| + |v_t||\nabla u| + |v||\nabla u_t|\big)|u_t| + |\rho_t|\big(|f| + |v||\nabla u|\big)|u_t| + |p_t|^2\big]\,\mathrm{d}x.
\end{aligned}$$

Thanks to the estimates (2.11), (2.12) and (2.16), we can easily show that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left|\sqrt{\rho}\,u_t\right|_{L^2}^2 + (2C)^{-1}|\nabla u_t|_{L^2}^2\\
&\quad\leqslant \widetilde{C}\big(1 + |f|_{L^3}^2 + |f_t|_{H^{-1}}^2 + |\nabla v_t|_{L^2}^2 + \left|\sqrt{\rho}\,u_t\right|_{L^2}^2\big).
\end{aligned}\tag{2.18}$$

Now we fix τ in $(0,T)$. Since the right-hand side of (2.18) is integrable in $(0,T)$, we deduce that

$$\left|\sqrt{\rho}\,u_t(t)\right|_{L^2}^2+\int_\tau^t |\nabla u_t|_{L^2}^2 \leqslant \widetilde{C}+\widetilde{C}\left|\sqrt{\rho}\,u_t(\tau)\right|_{L^2}^2 \quad \text{for } \tau \leqslant t \leqslant T. \tag{2.19}$$

To estimate $|\sqrt{\rho}\,u_t(\tau)|_{L^2}^2$, we observe from (2.2) that

$$\int \rho|u_t|^2\,\mathrm{d}x \leqslant 4\int \left(\rho|v|^2|\nabla u|^2+\rho|f|^2+\rho^{-1}|Lu+\nabla p|^2\right)\mathrm{d}x$$

and thus

$$\limsup_{\tau\to 0}\left|\sqrt{\rho}\,u_t(\tau)\right|_{L^2}^2 \leqslant \widetilde{C}\left(1+\mathcal{C}(\rho_0,u_0)\right),$$

where the functional $\mathcal{C}$ is defined by:

$$\mathcal{C}(\rho_0,u_0)=\int \rho_0^{-1}\left|Lu_0+\nabla\left(p(\rho_0)\right)\right|^2\mathrm{d}x. \tag{2.20}$$

Therefore, letting $\tau\to 0$ in (2.19), we conclude that

$$\sup_{0\leqslant t\leqslant T}\left|\sqrt{\rho}\,u_t(t)\right|_{L^2}^2+\int_0^T \left|u_t(t)\right|_{D_0^1}^2\,\mathrm{d}t \leqslant \widetilde{C}\left(1+\mathcal{C}(\rho_0,u_0)\right). \tag{2.21}$$

To obtain further estimates, observe that since for each $t\in[0,T]$, $u=u(t)\in D_0^1\cap D^2$ is a solution of the strongly elliptic system:

$$Lu=F \quad \text{with } F=\rho(f-u_t-v\cdot\nabla u)-\nabla p\in L^2\cap L^{q_0},$$

it follows from the elliptic regularity results in Section 5 that

$$|u|_{D^{2,r}} \leqslant C\left|\rho(f-u_t-v\cdot\nabla u)-\nabla p\right|_{L^r}+C|u|_{D^{1,r}} \quad (r=2,q_0). \tag{2.22}$$

It should be noted that the constant C in (2.22) is independent of the size of the domain Ω when Ω is the intersection of an unbounded domain and a large ball.

Therefore, using the previous estimates, we easily deduce from (2.22) that

$$\sup_{0\leqslant t\leqslant T}\left|u(t)\right|_{D^2}^2+\int_0^T \left|u(t)\right|_{D^{2,q_0}}^2\,\mathrm{d}t \leqslant \widetilde{C}\left(1+\mathcal{C}(\rho_0,u_0)\right). \tag{2.23}$$

We are now ready to prove the existence for the case of bounded domains. First, using standard regularization techniques, we choose $p^\delta=p^\delta(\cdot)$ and v^δ, $0<\delta\ll 1$, so that

$$p^\delta(\cdot)\in C^2[0,\infty),\quad p^\delta\to p\quad\text{in } C^1[0,\infty),$$
$$v^\delta\in L^\infty\big(0,T;D_0^1\cap D^2\big)\cap L^2\big(0,T;D^3\big),\quad v_t^\delta\in L^2\big(0,T;D_0^1\big)\quad\text{and}$$
$$\big(v^\delta,v_t^\delta\big)\to(v,v_t)\quad\text{in } L^\infty\big(0,T;D_0^1\cap D^2\big)\cap L^2\big(0,T;D^{2,q_0}\big)\times L^2\big(0,T;D_0^1\big).$$

Then, for each $\delta\in(0,1)$, let $\rho_0^\delta=\rho_0+\delta$ and let $u_0^\delta\in D_0^1\cap D^2$ is the solution to the boundary value problem $Lu_0^\delta=-\nabla p^\delta(\rho_0^\delta)+(\rho_0^\delta)^{1/2}g$ in Ω. It follows from the elliptic regularity result that $u_0^\delta\to u_0$ in $D_0^1\cap D^2$ as $\delta\to 0$. Hence, if we denote by (ρ^δ,u^δ) the solution of (2.1), (2.2) with the initial data $(\rho_0^\delta,u_0^\delta)$ and (p,v) replaced by (p^δ,v^δ), it satisfies the estimates (2.11), (2.12), (2.16), (2.21) and (2.23), where $\mathcal{C}(\rho_0^\delta,u_0^\delta)=|g|_{L^2}^2\leqslant\widetilde{C}$. Therefore, we conclude that a subsequence of solutions (ρ^δ,u^δ) converges to a limit (ρ,u) in a weak sense. It is easy to show that (ρ,u) is a weak solution to the original problem (2.1), (2.2), (1.2). Moreover, thanks to the lower semi-continuity of various norms, we have the following regularity estimate for (ρ,u):

$$\operatorname*{ess\,sup}_{0\leqslant t\leqslant T}\big(|\rho|_{H^1\cap W^{1,q_0}}+|\rho_t|_{L^2\cap L^{q_0}}+|u|_{D_0^1\cap D^2}+|\sqrt{\rho}\,u_t|_{L^2}\big)$$
$$+\int_0^T\big(|u|_{D^{2,q_0}}^2+|u_t|_{D_0^1}^2\big)\,\mathrm{d}t\leqslant\widetilde{C}\big(1+\mathcal{C}(\rho_0,u_0)\big).\tag{2.24}$$

2.2. Existence for unbounded domains

Next, we prove the existence for the case of unbounded domains; let Ω be the whole space $\mathbb{R}^3$, the half space $\mathbb{R}^3_+$ or an exterior domain in $\mathbb{R}^3$. For a fixed large $R>0$, let $u_0^R\in D_0^1(\Omega_R)\cap D^2(\Omega_R)$ be a unique solution to the boundary value problem:

$$Lu_0^R=-\nabla p(\rho_0)+\rho_0^{1/2}g\quad\text{in }\Omega_R=\Omega\cap B_R,\tag{2.25}$$

where B_R is the open ball of radius R centered at 0. Then we extend u_0^R to Ω by defining 0 outside Ω_R. We claim that $u_0^R\to u_0$ in $D_0^1(\Omega)$ as $R\to\infty$. To show this, note first that $Lu_0^R=Lu_0$ in Ω_R. Thus multiplying this by u_0^R and integrating over Ω_R, we obtain:

$$\int\mu\big|\nabla u_0^R\big|^2+(\lambda+\mu)\big(\operatorname{div}u_0^R\big)^2\,\mathrm{d}x$$
$$=\int\mu\nabla u_0\cdot\nabla u_0^R+(\lambda+\mu)\operatorname{div}u_0\operatorname{div}u_0^R\,\mathrm{d}x.\tag{2.26}$$

In particular, it follows that $|u_0^R|_{D_0^1(\Omega)}\leqslant C|u_0|_{D_0^1(\Omega)}$. Hence there exists a sequence (R_j), $R_j\to\infty$, such that $u_0^{R_j}$ converges weakly in $D_0^1(\Omega)$ to a limit w. Then $w\in D_0^1(\Omega)$ is a weak solution of the equations $Lw=-\nabla p(\rho_0)+\rho_0^{1/2}g$ in Ω. In view of the uniqueness in $D_0^1(\Omega)$, we deduce that $w=u_0$ in Ω. Moreover, this implies that $u_0^R\rightharpoonup u_0$ in $D_0^1(\Omega)$ as

$R \to \infty$, since any weakly convergent subsequence of (u_0^R) in $D_0^1(\Omega)$ converges weakly to the same limit u_0. Then, the strong convergence of (u_0^R) to u_0 follows from the weak convergence and the identity (2.26).

Now let (ρ^R, u^R) be a strong solution of (2.1), (2.2) in Ω_R with the initial data (ρ_0, u_0^R), constructed in Section 2.1. Then since $|u_0^R|_{D_0^1(\Omega)} \leqslant \widetilde{C}$ and (ρ_0, u_0^R) satisfies the compatibility condition (2.25), we conclude that (ρ^R, u^R) also satisfies the estimate (2.24) in Ω_R. Hence, if we extend (ρ^R, u^R) to Ω by defining 0 outside Ω_R, then there exists a sequence (R_j), $R_j \to \infty$, such that (ρ^{R_j}, u^{R_j}) converges to a limit (ρ, u) in the following weak sense:

$$u^{R_j} \overset{*}{\rightharpoonup} u \quad \text{in } L^\infty\big(0, T; D_0^1(\Omega) \cap H^2_{\mathrm{loc}}(\Omega)\big),$$

$$\rho^{R_j} \overset{*}{\rightharpoonup} \rho \quad \text{in } L^\infty\big(0, T; W^{1,q_0}_{\mathrm{loc}}(\Omega)\big), \qquad u_t^{R_j} \rightharpoonup u_t \quad \text{in } L^2\big(0, T; D_0^1(\Omega)\big).$$

Moreover, (ρ, u) also satisfies the regularity estimate (2.24). Therefore, recalling that $u_0^R \to u_0$ in $D_0^1(\Omega)$, we easily show that (ρ, u) is a weak solution to the original problem (2.1), (2.2), (1.2) satisfying the regularity (2.5) except for the continuity.

2.3. *Continuity and uniqueness*

To complete the proof of the theorem, it remains to prove the continuity and uniqueness of the strong solutions constructed in the previous sections.

We first prove the continuity of the solution (ρ, u). The continuity of ρ can be proved by a standard argument from the theory of hyperbolic equations. Since ρ satisfies the regularity (2.24), it follows from a result of R.J. DiPerna and P.L. Lions [6] and classical embedding results (see R. Teman [22], for instance) that

$$\rho \in C\big([0, T]; L^2 \cap L^{q_0}\big) \cap C\big([0, T]; H^1 \cap W^{1,q_0}\text{-weak}\big).$$

To show the strong continuity in $H^1 \cap W^{1,q_0}$, observe from (2.10) and (2.11) that for $r = 2, q_0$ and $j = 1, 2, 3$,

$$\big|\partial_j \rho(t)\big|_{L^r}^r \leqslant \big|\partial_j \rho(0)\big|_{L^r}^r + \widetilde{C} \int_0^t \big|\nabla v(s)\big|_{H^1 \cap D^{1,q_0}} \,\mathrm{d}s$$

and thus $\limsup_{t \to +0} |\partial_j \rho(t)|_{L^r}^r \leqslant |\partial_j \rho(0)|_{L^r}^r$. Hence it follows from a well-known criterion on the strong convergence for the space L^r (see G. Galdi [8], for instance) that

$$\lim_{t \to +0} \big|\partial_j \rho(t) - \partial_j \rho(0)\big|_{L^r}^r = 0.$$

Therefore, the continuity of $\nabla \rho$ in L^r $(r = 2, q_0)$ follows from this result and the observation that for each fixed $t_0 \in [0, T]$, the function $\tilde{\rho} = \tilde{\rho}(t, x) = \rho(\pm t + t_0, x)$ is a unique strong solution to the similar initial value problem

$$\tilde{\rho}_t + \operatorname{div}(\tilde{\rho}\tilde{v}) = 0 \quad \text{and} \quad \tilde{\rho}(0) = \rho(t_0),$$

where $\tilde{v} = \tilde{v}(t,x) = \pm v(\pm t + t_0, x)$.

To show the continuity of u, we first observe that

$$u, v \in C\big([0,T]; D_0^1\big) \cap C\big([0,T]; D^2\text{-weak}\big).$$

We then prove the continuity of ρu_t in L^2. From the linearized momentum equation (2.2), we can easily deduce (see the proof of Remark 4 below) that $(\rho u_t)_t \in L^2(0,T; H^{-1})$, where H^{-1} is the dual space of H_0^1. Then since $\rho u_t \in L^2(0,T; H_0^1)$, it follows from a standard embedding result that $\rho u_t \in C([0,T]; L^2)$. Therefore, we conclude that for each $t \in [0,T]$, $u = u(t) \in D_0^1 \cap D^2$ is a solution of the elliptic system:

$$Lu = G - \rho v \cdot \nabla u, \quad \text{where } G = \rho f - \rho u_t - \nabla p(\rho) \in C\big([0,T]; L^2\big).$$

Now it is not difficult to show that $u \in C([0,T]; D^2)$. In view of the elliptic regularity estimate (2.22), we obtain:

$$\begin{aligned} \big|u(t) - u(s)\big|_{D^2} \leqslant C\big|u(t) - u(s)\big|_{D_0^1} + C\big|G(t) - G(s)\big|_{L^2} \\ + C\big|\rho v \cdot \nabla u(t) - \rho v \cdot \nabla u(s)\big|_{L^2}. \end{aligned} \tag{2.27}$$

Using the estimate (2.24), we obtain:

$$\begin{aligned} &C\big|\rho v \cdot \nabla u(t) - \rho v \cdot \nabla u(s)\big|_{L^2} \\ &\quad \leqslant C\big|\big(\rho(t) - \rho(s)\big)v(t) \cdot \nabla u(t)\big|_{L^2} + C\big|\rho(s)\big(v(t) - v(s)\big) \cdot \nabla u(t)\big|_{L^2} \\ &\qquad + C\big|\rho(s)v(s) \cdot \big(\nabla u(t) - \nabla u(s)\big)\big|_{L^2} \\ &\quad \leqslant C\big(\big|\rho(t) - \rho(s)\big|_{L^\infty}\big|\nabla v(t)\big|_{L^2} + \big|\rho(s)\big|_{L^\infty}\big|\nabla\big(v(t) - v(s)\big)\big|_{L^2}\big)\big|\nabla u(t)\big|_{L^3} \\ &\qquad + C\big|\rho(s)\big|_{L^\infty}\big|\nabla v(s)\big|_{L^2}\big|\nabla\big(u(t) - u(s)\big)\big|_{L^2}^{1/2}\big|\nabla\big(u(t) - u(s)\big)\big|_{H^1}^{1/2} \\ &\quad \leqslant \widetilde{C}\big(\big|\rho(t) - \rho(s)\big|_{L^\infty} + \big|v(t) - v(s)\big|_{D_0^1} + \big|u(t) - u(s)\big|_{D_0^1}\big) + \frac{1}{2}\big|u(t) - u(s)\big|_{D^2}. \end{aligned}$$

Substituting this into (2.27), we conclude that $|u(t) - u(s)|_{D^2} \leqslant \Theta(t,s)$ for some function $\Theta(t,s)$ such that $\lim_{t \to s} \Theta(t,s) = 0$. Hence the continuity of u in D^2 follows. This completes the proof of the continuity.

Finally, we prove the uniqueness of solutions satisfying the strong regularity (2.24). Let (ρ_1, u_1) and (ρ_2, u_2) be two strong solutions to the problem (2.1), (2.2) and (1.2). Denote $\bar{\rho} = \rho_1 - \rho_2$ and $\bar{u} = u_1 - u_2$. Then it follows from (2.1) that $\frac{\mathrm{d}}{\mathrm{d}t}\int |\bar{\rho}|^2\,\mathrm{d}x \leqslant \int |\operatorname{div} v||\bar{\rho}|^2\,\mathrm{d}x$. Since $\nabla v \in L^2(0,T; W^{1,q_0})$ and $\bar{\rho}(0) = 0$, we deduce from Gronwall's inequality that $|\bar{\rho}|_{L^2} = 0$, that is, $\rho_1 = \rho_2$ in $(0,T) \times \Omega$. Next, we choose a cut-off function $\varphi \in C_c^\infty(\mathbb{R}^3)$ such that $\varphi(x) = 1$ for $|x| \leqslant 1$ and $\varphi(x) = 0$ for $|x| \geqslant 2$, and define $\varphi_R(x) = \varphi(x/R)$ for

$x \in \mathbb{R}^3$. Then multiplying the equation $\rho_1 \bar{u}_t + \rho_1 v \cdot \nabla \bar{u} + L\bar{u} = 0$ by $\varphi_R^2 \bar{u}$, integrating over $(0, T) \times \Omega$ and letting $R \to \infty$, we easily derive:

$$\frac{1}{2}\int \rho_1 |\bar{u}|^2(t)\,\mathrm{d}x + \int_0^t \int \big(\mu|\nabla\bar{u}|^2 + (\lambda+\mu)(\operatorname{div}\bar{u})^2\big)\,\mathrm{d}x\,\mathrm{d}s = \frac{1}{2}\int \rho_1|\bar{u}|^2(0)\,\mathrm{d}x.$$

Hence recalling again that $\bar{u}(0) = 0$, we deduce that $|\rho_1^{1/2}\bar{u}|_{L^2} = 0$ and $|\nabla\bar{u}|_{L^2} = 0$ in $(0, T)$. Therefore, we conclude that $\bar{u} = 0$ in $(0, T) \times \Omega$, because $\bar{u} \in C([0, T]; D_0^1)$. This completes the proof of the theorem.

Remark 4. If (ρ, u) is a strong solution of the linearized system (2.1) and (2.2) satisfying the regularity (2.24), then it satisfies the following identity: for a.e. $t \in (0, T)$,

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int \rho|u_t|^2\,\mathrm{d}x + \int \mu|\nabla u_t|^2 + (\lambda+\mu)(\operatorname{div} u_t)^2\,\mathrm{d}x - \int p_t \operatorname{div} u_t\,\mathrm{d}x \\ = \int \big(\rho f_t + \rho_t(f - u_t - v\cdot\nabla u) - \rho v_t \cdot \nabla u\big)\cdot u_t\,\mathrm{d}x.\end{aligned} \tag{2.28}$$

Proof. For a.e. $t \in (0, T)$ and all $w \in H_0^1$,

$$\begin{aligned}(\rho u_t, w)_{L^2} &= \big(-\rho v\cdot\nabla u + \mu\Delta u + (\lambda+\mu)\nabla\operatorname{div} u - \nabla p + \rho f, w\big)_{L^2} \\ &= (-\rho v\cdot\nabla u + \rho f, w)_{L^2} - (\mu\nabla u, \nabla w)_{L^2} - \big((\lambda+\mu)\operatorname{div} u - p, \operatorname{div} w\big)_{L^2}\end{aligned}$$

and thus

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\rho u_t, w)_{L^2} = \big((-\rho v\cdot\nabla u + \rho f)_t, w\big)_{L^2} - (\mu\nabla u_t, \nabla w)_{L^2} \\ - \big((\lambda+\mu)\operatorname{div} u_t - p_t, \operatorname{div} w\big)_{L^2}.\end{aligned} \tag{2.29}$$

Using the regularity (2.24) of (ρ, u), we show that the right-hand side of (2.29) is bounded above by $A(t)|w|_{H_0^1}$ for some positive function $A(t) \in L^2(0, T)$. Hence it follows, from the well-known result (see Lemma 1.1, Chapter 3 in [22]) that $(\rho u_t)_t \in L^2(0, T; H^{-1})$ and $\frac{\mathrm{d}}{\mathrm{d}t}(\rho u_t, w)_{L^2} = \langle(\rho u_t)_t, w\rangle$ for all $w \in H_0^1$. Here H^{-1} denotes the dual space of H_0^1 and $\langle\cdot,\cdot\rangle$ the corresponding dual pairing. Hence if Ω is a bounded domain, then since $D_0^1 = H_0^1$, the identity (2.28) follows easily from:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \rho|u_t|^2\,\mathrm{d}x = 2\langle(\rho u_t)_t, u_t\rangle - \int \rho_t|u_t|^2\,\mathrm{d}x. \tag{2.30}$$

To prove it for unbounded domain, we define $w_R \in L^2(0, T_*; H_0^1(\Omega))$ by:

$$w_R(t, x) = \varphi_R(x)u_t(t, x) \quad \text{for } (t, x) \in (0, T_*) \times \Omega,$$

where φ_R is the cut-off function defined in Section 2.3. Then we deduce from (2.29) and (2.30) that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int \rho|w_R|^2\,\mathrm{d}x + \int \varphi_R^2\big(\mu|\nabla u_t|^2 + (\lambda+\mu)(\operatorname{div} u_t)^2\big)\,\mathrm{d}x$$
$$= \int \varphi_R\big((-\rho v\cdot\nabla u + \rho f)_t\big)\cdot w_R\,\mathrm{d}x + \int \varphi_R^2 p_t \operatorname{div} u_t\,\mathrm{d}x + \mathcal{E}_R(t), \tag{2.31}$$

where the remainder term $\mathcal{E}_R(t)$ satisfies:

$$\int_0^T \mathcal{E}_R(t)\,\mathrm{d}t \leqslant \frac{C}{R}\int_0^T \int_{B_{2R}\setminus B_R} \big(|u_t||\nabla u_t| + |\rho_t||u_t|\big)\,\mathrm{d}x\,\mathrm{d}t \to 0$$

as $R\to\infty$. Hence integrating (2.31) over (τ, t) and letting $R\to\infty$, we easily prove the integral form of the identity (2.28). □

3. A local existence result for positive densities

In this section, we assume that Ω is a *bounded domain* in $\mathbb{R}^3$ with smooth boundary and then prove a local existence result on strong solutions with positive densities to the original nonlinear problem (1.1), (1.2). Furthermore, we derive some uniform bounds which are independent of the lower bounds of the initial density and the size of the domain. The bounds will be used in the next section to prove the existence of strong solutions with nonnegative densities.

Proposition 5. *Assume that* $p = p(\cdot) \in C^1[0,\infty)$, *and the data* (ρ_0, u_0, f) *satisfy the regularity conditions*:

$$\rho_0 \in H^1\cap W^{1,q}, \quad u_0 \in D_0^1\cap D^2,$$
$$f \in C\big([0,T]; L^2\big)\cap L^2\big(0,T; L^q\big) \quad \textit{and} \quad f_t \in L^2\big(0,T; H^{-1}\big) \tag{3.1}$$

for some q *with* $3<q<\infty$. *Assume further that* $\rho_0 \geqslant \delta$ *in* Ω *for some constant* $\delta>0$. *Then there exist a time* $T_* \in (0,T)$ *and a unique strong solution* (ρ, u) *to the nonlinear problem* (1.1), (1.2) *such that*

$$\rho \in C\big([0,T_*]; H^1\cap W^{1,q_0}\big), \quad u \in C\big([0,T_*]; D_0^1\cap D^2\big)\cap L^2\big(0,T_*; D^{2,q_0}\big),$$
$$\rho_t \in C\big([0,T_*]; L^2\cap L^{q_0}\big), \quad u_t \in L^2\big(0,T_*; D_0^1\big) \quad \textit{and} \quad \sqrt{\rho}\,u_t \in L^\infty\big(0,T_*; L^2\big),$$

where $q_0 = \min(6,q)$. *Furthermore, we have the following estimates*:

$$\operatorname*{ess\,sup}_{0<t<T_*}\left(\left|\rho(t)\right|_{H^1\cap W^{1,q_0}}+\left|\rho_t(t)\right|_{L^2\cap L^{q_0}}+\left|u(t)\right|_{D_0^1\cap D^2}+\left|\sqrt{\rho}\,u_t(t)\right|_{L^2}\right)$$
$$+\int_0^{T_*}\left(\left|u(t)\right|^2_{D^{2,q_0}}+\left|u_t(t)\right|^2_{D_0^1}\right)\mathrm{d}t\leqslant C\exp\left[C\exp(C\mathcal{C}_0)\right],\tag{3.2}$$

where

$$\mathcal{C}_0=\mathcal{C}(\rho_0,u_0)=\int\rho_0^{-1}\left|Lu_0+\nabla p(\rho_0)\right|^2\mathrm{d}x.\tag{3.3}$$

The constant C and the local existence time T_* in (3.2) *are independent of* δ.

To prove the proposition, we first construct approximate solutions, inductively, as follows:

(i) first define $u^0=0$, and
(ii) assuming that u^{k-1} was defined for $k\geqslant 1$, let (ρ^k,u^k) be the unique solution to the following initial boundary value problem:

$$\rho_t^k+u^{k-1}\cdot\nabla\rho^k+\rho^k\operatorname{div}u^{k-1}=0\quad\text{in }(0,T)\times\Omega,\tag{3.4}$$
$$\rho^k u_t^k+\rho^k u^{k-1}\cdot\nabla u^k+Lu^k+\nabla p^k=\rho^k f\quad\text{in }(0,T)\times\Omega,\tag{3.5}$$
$$\rho^k|_{t=0}=\rho_0,\quad u^k|_{t=0}=u_0\quad\text{in }\Omega;\qquad u^k=0\quad\text{on }(0,T)\times\partial\Omega,\tag{3.6}$$
$$\rho^k(t,x)\to 0,\quad u^k(t,x)\to 0\quad\text{as }|x|\to\infty,\ (t,x)\in(0,T)\times\Omega,\tag{3.7}$$

where $Lu^k=-\mu\Delta u^k-(\lambda+\mu)\nabla\operatorname{div}u^k$ and $p^k=p(\rho^k)$. The existence of a global strong solution (ρ^k,u^k) with the regularity (2.5) to the linearized problem (3.4)–(3.6) was proved in the previous section. Note also that $\rho^k>0$ on $[0,T]\times\overline{\Omega}$ (recall (2.7) and (2.8)).

From now on, we derive uniform bounds on the approximate solutions and then prove the convergence of the approximate solutions to a strong solution of the original nonlinear problem.

3.1. Uniform bounds

Let $K\geqslant 1$ be a fixed large integer, and let us introduce an auxiliary function $\Phi_K(t)$, defined by:

$$\Phi_K(t)=\max_{1\leqslant k\leqslant K}\sup_{0\leqslant s\leqslant t}\left(1+\left|\rho^k(s)\right|_{H^1\cap W^{1,q_0}}+\left|u^k(s)\right|_{D_0^1}\right).$$

Then we estimate each term of Φ_K in terms of some integrals of Φ_K, apply arguments of Gronwall-type and thus prove that Φ_K is locally bounded.

We first estimate the second term $|u^k|_{D_0^1}$ for $1 \leqslant k \leqslant K$. Multiplying (3.5) by $\varphi_R u_t^k$, where φ_R is the cut-off function as in the previous section, and letting $R \to \infty$, we obtain the analogue of (2.15):

$$\begin{aligned}
&\frac{1}{2}\int \rho^k |u_t^k|^2 \,\mathrm{d}x + \frac{\mathrm{d}}{\mathrm{d}t}\int \frac{\mu}{2}|\nabla u^k|^2 + \frac{\lambda+\mu}{2}\big(\operatorname{div} u^k\big)^2 \,\mathrm{d}x \\
&\quad \leqslant \frac{\mathrm{d}}{\mathrm{d}t}\int p^k \operatorname{div} u^k \,\mathrm{d}x + \int \rho^k |f|^2 + \rho^k \big|u^{k-1}\big|^2 \big|\nabla u^k\big|^2 + \big|p_t^k\big|\big|\nabla u^k\big| \,\mathrm{d}x. \qquad (3.8)
\end{aligned}$$

Integrating this over $(0,t)$ and using (3.4), we deduce that

$$\begin{aligned}
&\int_0^t \int \rho^k |u_t^k|^2 \,\mathrm{d}x\,\mathrm{d}s + \int \big|\nabla u^k(t)\big|^2 \,\mathrm{d}x \\
&\quad \leqslant C + C\int p^k(t)^2 \,\mathrm{d}x + C\int_0^t \Big(\big|\rho^k\big|_{L^\infty} + \big|\rho^k\big|_{L^\infty}\big|u^{k-1}\big|_{L^6}^2 \big|\nabla u^k\big|_{L^3}^2 \\
&\qquad + \big|\nabla p^k\big|_{L^3}\big|u^{k-1}\big|_{L^6}\big|\nabla u^k\big|_{L^2} + \big|p^k\big|_{L^\infty}\big|\nabla u^{k-1}\big|_{L^2}\big|\nabla u^k\big|_{L^2}\Big) \,\mathrm{d}s. \qquad (3.9)
\end{aligned}$$

To estimate the right-hand side of (3.9), we first observe that

$$\big|\rho^k\big|_{L^\infty} + \big|p^k\big|_{H^1\cap W^{1,q_0}} \leqslant C\Phi_K + C\big|p'\big(\rho^k\big)\big|_{L^\infty}\Phi_K \leqslant M(\Phi_K) \qquad (3.10)$$

for some increasing continuous function $M = M(\cdot):[0,\infty) \to [0,\infty)$ with $M(0) = 0$. Hence it follows from (3.4) and Sobolev inequality that

$$\begin{aligned}
\int p^k(t)^2 \,\mathrm{d}x &= \int p^k(0)^2 \,\mathrm{d}x + \int_0^t \frac{\partial}{\partial s}\left(\int p^k(s)^2 \,\mathrm{d}x\right)\mathrm{d}s \\
&= \int p^k(0)^2 \,\mathrm{d}x + 2\int_0^t \int p'\big(\rho^k\big)p^k\big(-\nabla\rho^k \cdot u^{k-1} - \rho^k \operatorname{div} u^{k-1}\big)\,\mathrm{d}x\,\mathrm{d}s \\
&\leqslant C + C\int_0^t \big|p'\big(\rho^k\big)\big|_{L^\infty}\big(\big|p^k\big|_{L^3}\big|\nabla\rho^k\big|_{L^2} + \big|p^k\big|_{L^2}\big|\rho^k\big|_{L^\infty}\big)\big|\nabla u^{k-1}\big|_{L^2} \,\mathrm{d}s \\
&\leqslant C + \int_0^t M\big(\Phi_K(s)\big)\,\mathrm{d}s
\end{aligned}$$

for some function $M = M(\cdot)$. Sobolev inequality also yields:

$$\left|\nabla u^k\right|_{L^3} \leqslant \left|\nabla u^k\right|_{L^2}^{1/2}\left|\nabla u^k\right|_{L^6}^{1/2} \leqslant C\left|\nabla u^k\right|_{L^2}^{1/2}\left|\nabla u^k\right|_{H^1}^{1/2}.$$

Substituting these estimates into (3.9), we thus have:

$$\int_0^t \left|\sqrt{\rho^k}\, u_t^k\right|_{L^2}^2 \mathrm{d}s + \left|\nabla u^k(t)\right|_{L^2}^2 \leqslant C + \int_0^t M(\Phi_K)\left(1 + \left|\nabla u^k\right|_{H^1}\right) \mathrm{d}s. \tag{3.11}$$

Throughout the paper, we denote by $M = M(\cdot)$ an increasing continuous function from $[0, \infty)$ to itself with $M(0) = 0$, which is independent of δ and the size of Ω.

To estimate the higher order term $|\nabla u^k|_{H^1}$, observe that for any $t \in [0, T]$, $u^k = u^k(t) \in D_0^1 \cap D^2$ is a solution of the elliptic system $Lu^k = F^k$ in Ω, where $F^k = \rho^k f - \rho^k u_t^k - \rho^k u^{k-1} \cdot \nabla u^k - \nabla p^k$. Hence we deduce from the elliptic regularity result that

$$\begin{aligned}
\left|\nabla^2 u^k\right|_{L^2} &\leqslant C\left(\left|\rho^k f\right|_{L^2} + \left|\rho^k u_t^k\right|_{L^2} + \left|\rho^k u^{k-1} \cdot \nabla u^k\right|_{L^2} + \left|\nabla p^k\right|_{L^2} + \left|\nabla u^k\right|_{L^2}\right) \\
&\leqslant M(\Phi_K)\left(1 + \left|\sqrt{\rho^k}\, u_t^k\right|_{L^2}\right) + C\left|\rho^k\right|_{L^\infty}\left|\nabla u^{k-1}\right|_{L^2}\left|\nabla u^k\right|_{L^3} \\
&\leqslant M(\Phi_K)\left(1 + \left|\sqrt{\rho^k}\, u_t^k\right|_{L^2}\right) + \frac{1}{2}\left|\nabla u^k\right|_{H^1}
\end{aligned}$$

and thus

$$\left|\nabla u^k\right|_{H^1} \leqslant M(\Phi_K)\left(1 + \left|\sqrt{\rho^k}\, u_t^k\right|_{L^2}\right). \tag{3.12}$$

Substituting this into (3.11) and using Young's inequality, we conclude that

$$\int_0^t \left|\sqrt{\rho^k}\, u_t^k(s)\right|_{L^2}^2 \mathrm{d}s + \left|u^k(t)\right|_{D_0^1}^2 \leqslant C + \int_0^t M\left(\Phi_K(s)\right) \mathrm{d}s \tag{3.13}$$

for all k, $1 \leqslant k \leqslant K$.

Next we estimate $|\sqrt{\rho^k}\, u_t^k|_{L^2}$ and $|u_t^k|_{D_0^1}$ to derive the higher regularity estimates. To begin with we recall from Remark 4 that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int \rho^k\left|u_t^k\right|^2 \mathrm{d}x + \int \mu\left|\nabla u_t^k\right|^2 + (\lambda + \mu)\left(\operatorname{div} u_t^k\right)^2 \mathrm{d}x - \int p_t^k \operatorname{div} u_t^k \,\mathrm{d}x \\
&\quad = \int \left(\operatorname{div}\left(\rho^k u^{k-1}\right)\left(u_t^k + u^{k-1} \cdot \nabla u^k - f\right) - \rho^k u_t^{k-1} \cdot \nabla u^k + \rho^k f_t\right) \cdot u_t^k \,\mathrm{d}x.
\end{aligned}$$

Using the linearized continuity equation (3.4), we deduce that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int \frac{1}{2}\rho^k\left|u_t^k\right|^2\mathrm{d}x+\mu\int\left|\nabla u_t^k\right|^2\mathrm{d}x\\
&\quad\leqslant\int\big(2\rho^k\left|u^{k-1}\right|\left|u_t^k\right|\left|\nabla u_t^k\right|+\rho^k\left|u^{k-1}\right|\left|\nabla u^{k-1}\right|\left|\nabla u^k\right|\left|u_t^k\right|\\
&\qquad+\rho^k\left|u^{k-1}\right|^2\left|u_t^k\right|\left|\nabla^2 u^k\right|+\rho^k\left|u^{k-1}\right|^2\left|\nabla u^k\right|\left|\nabla u_t^k\right|\\
&\qquad+\rho^k\left|u_t^{k-1}\right|\left|u_t^k\right|\left|\nabla u^k\right|+\left|\nabla p^k\right|\left|u^{k-1}\right|\left|\operatorname{div} u_t^k\right|\\
&\qquad+p'\big(\rho^k\big)\rho^k\left|\operatorname{div} u^{k-1}\right|\left|\operatorname{div} u_t^k\right|+\left|\nabla\rho^k\right|\left|u^{k-1}\right||f|\left|u_t^k\right|\\
&\qquad+\rho^k\left|\nabla u^{k-1}\right||f|\left|u_t^k\right|+\rho^k\left|u^{k-1}\right||f|\left|\nabla u_t^k\right|+\rho^k u_t^k\cdot f_t\big)\,\mathrm{d}x\equiv\sum_{j=1}^{11}I_j. \quad (3.14)
\end{aligned}$$

To simplify the estimation of each term I_j, we introduce:

$$A_K(t):=\max_{1\leqslant k\leqslant K}\big(1+\left|\sqrt{\rho^k}\,u_t^k\right|_{L^2}^2\big).$$

In the following calculations, we will make extensive use of Sobolev inequality, Hölder inequality, (3.10) and (3.12). Let

$$\begin{aligned}
I_1&\leqslant 2\left|\rho^k\right|_{L^\infty}^{1/2}\left|u^{k-1}\right|_{L^6}\left|\sqrt{\rho^k}\,u_t^k\right|_{L^3}\left|\nabla u_t^k\right|_{L^2}\leqslant 2\Phi_K^{7/4}\left|\sqrt{\rho^k}\,u_t^k\right|_{L^2}^{1/2}\left|u_t^k\right|_{L^6}^{1/2}\left|\nabla u_t^k\right|_{L^2}\\
&\leqslant C\Phi_K^{7/4}\left|\sqrt{\rho^k}\,u_t^k\right|_{L^2}^{1/2}\left|\nabla u_t^k\right|_{L^2}^{3/2}\leqslant C_\varepsilon\Phi_K^7A_K+\varepsilon\left|\nabla u_t^k\right|_{L^2}^2,\\
I_2&\leqslant\left|\rho^k\right|_{L^\infty}\left|u^{k-1}\right|_{L^6}\left|\nabla u^{k-1}\right|_{L^3}\left|\nabla u^k\right|_{L^3}\left|u_t^k\right|_{L^6}\\
&\leqslant C\Phi_K^3\left|\nabla u^{k-1}\right|_{H^1}^{1/2}\left|\nabla u^k\right|_{H^1}^{1/2}\left|\nabla u_t^k\right|_{L^2}\leqslant C_\varepsilon M(\Phi_K)A_K+\varepsilon\left|\nabla u_t^k\right|_{L^2}^2,\\
I_3&\leqslant\left|\rho^k\right|_{L^\infty}\left|u^{k-1}\right|_{L^6}^2\left|u_t^k\right|_{L^6}\left|\nabla^2u^k\right|_{L^2}\leqslant C\Phi_K^3\left|\nabla u_t^k\right|_{L^2}\left|\nabla u^k\right|_{H^1}\\
&\leqslant C_\varepsilon M(\Phi_K)A_K+\varepsilon\left|\nabla u_t^k\right|_{L^2}^2,\\
I_4&\leqslant\left|\rho^k\right|_{L^\infty}\left|u^{k-1}\right|_{L^6}^2\left|\nabla u^k\right|_{L^6}\left|\nabla u_t^k\right|_{L^2}\leqslant C\Phi_K^3\left|\nabla u^k\right|_{H^1}\left|\nabla u_t^k\right|_{L^2}\\
&\leqslant C_\varepsilon M(\Phi_K)A_K+\varepsilon\left|\nabla u_t^k\right|_{L^2}^2,\\
I_5&\leqslant\left|\rho^k\right|_{L^\infty}^{1/2}\left|u_t^{k-1}\right|_{L^6}\left|\sqrt{\rho^k}\,u_t^k\right|_{L^3}\left|\nabla u^k\right|_{L^2}\leqslant C\Phi_K^{7/4}\left|\nabla u_t^{k-1}\right|_{L^2}\left|\sqrt{\rho^k}\,u_t^k\right|_{L^2}^{1/2}\left|\nabla u_t^k\right|_{L^2}^{1/2}\\
&\leqslant C_\eta\Phi_K^{7/2}\left|\sqrt{\rho^k}\,u_t^k\right|_{L^2}\left|\nabla u_t^k\right|_{L^2}+\eta\left|\nabla u_t^{k-1}\right|_{L^2}^2\\
&\leqslant C_{\eta,\varepsilon}\Phi_K^7A_K+\varepsilon\left|\nabla u_t^k\right|_{L^2}^2+\eta\left|\nabla u_t^{k-1}\right|_{L^2}^2,\\
I_6&\leqslant\left|\nabla p^k\right|_{L^3}\left|\nabla u^{k-1}\right|_{L^2}\left|\nabla u_t^k\right|_{L^2}\leqslant C_\varepsilon M(\Phi_K)+\varepsilon\left|\nabla u_t^k\right|_{L^2}^2,\\
I_7&\leqslant C\left|p'\big(\rho^k\big)\right|_{L^\infty}\left|\rho^k\right|_{L^\infty}\left|\nabla u^{k-1}\right|_{L^2}\left|\nabla u_t^k\right|_{L^2}\leqslant C_\varepsilon M(\Phi_K)+\varepsilon\left|\nabla u_t^k\right|_{L^2}^2,\\
I_8+I_9&\leqslant C\big(\left|\nabla\rho^k\right|_{L^3}+\left|\rho^k\right|_{L^\infty}\big)\left|\nabla u^{k-1}\right|_{L^2}|f|_{L^3}\left|\nabla u_t^k\right|_{L^2}\leqslant C_\varepsilon\Phi_K^4|f|_{L^3}^2+\varepsilon\left|\nabla u_t^k\right|_{L^2}^2,
\end{aligned}$$

$$I_{10} \leqslant C_\varepsilon \Phi_K^4 |f|_{L^3}^2 + \varepsilon \left|\nabla u_t^k\right|_{L^2}^2 \leqslant C_\varepsilon \Phi_K^8 + |f|_{L^3}^2 + \varepsilon \left|\nabla u_t^k\right|_{L^2}^2,$$

and finally

$$\begin{aligned} I_{11} &\leqslant C|f_t|_{H^{-1}}\left(\left|\rho^k\right|_{L^\infty}^{1/2}\left|\sqrt{\rho^k}\, u_t^k\right|_{L^2} + \left(\left|\nabla\rho^k\right|_{L^3} + \left|\rho^k\right|_{L^\infty}\right)\left|\nabla u_t^k\right|_{L^2}\right) \\ &\leqslant C_\varepsilon \Phi_K^2 |f_t|_{H^{-1}}^2 + A_K + \varepsilon\left|\nabla u^k\right|_{L^2}^2. \end{aligned}$$

Inserting all the estimates into (3.14) and choosing $\varepsilon, \eta > 0$ so small, we obtain:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\int \rho^k \left|u_t^k\right|^2 + \mu \int \left|\nabla u_t^k\right|^2 \mathrm{d}x &\leqslant CM(\Phi_K)A_K + C\Phi_K^4|f|_{L^3}^2 + C\Phi_K^2|f_t|_{H^{-1}}^2 \\ &\quad + \frac{\mu}{2}\left|\nabla u_t^{k-1}\right|_{L^2}^2. \end{aligned}$$

Hence integrating over $(\tau, t) \Subset (0, T)$, we have:

$$\begin{aligned} \left|\sqrt{\rho^k}\, u_t^k(t)\right|_{L^2}^2 + \mu \int_\tau^t \left|\nabla u_t^k\right|_{L^2}^2 \mathrm{d}s &\leqslant C\left(A_K(\tau) + \int_\tau^t M(\Phi_K)A_K\, \mathrm{d}s\right) \\ &\quad + \frac{\mu}{2}\int_\tau^t \left|\nabla u_t^{k-1}\right|_{L^2}^2 \mathrm{d}s. \end{aligned} \tag{3.15}$$

But, from the recursive relation of $|\nabla u_t^k|_{L^2}$, it follows that

$$\begin{aligned} \mu \int_\tau^t \left|\nabla u_t^k\right|_{L^2}^2 \mathrm{d}s &\leqslant \left(1 + \frac{1}{2} + \frac{1}{4} + \cdots\right) C\left(A_K(\tau) + \int_\tau^t M(\Phi_K)A_K\, \mathrm{d}s\right) \\ &\leqslant 2C\left(A_K(\tau) + \int_\tau^t M(\Phi_K)A_K\, \mathrm{d}s\right) \end{aligned}$$

for $1 \leqslant k \leqslant K$. Thus, we deduce from (3.15) that

$$\left|\sqrt{\rho^k}\, u_t^k(t)\right|_{L^2}^2 + \int_\tau^t \left|\nabla u_t^k\right|^2 \mathrm{d}s \leqslant C\left(A_K(\tau) + \int_\tau^t M(\Phi_K)A_K\, \mathrm{d}s\right) \tag{3.16}$$

for all k, $1 \leqslant k \leqslant K$. In particular, we have:

$$A_K(t) \leqslant CA_K(\tau) + C\int_\tau^t M\left(\Phi_K(s)\right)A_K(s)\, \mathrm{d}s.$$

Hence applying Gronwall's inequality, we obtain:

$$A_K(t) \leqslant C A_K(\tau) \exp\left(C \int_\tau^t M\big(\Phi_K(s)\big)\,\mathrm{d}s\right). \tag{3.17}$$

To estimate $\limsup_{\tau\to 0} A_K(\tau)$, we multiply Eq. (3.5) by u_t^k and integrate over Ω. Then we have:

$$\int \rho^k \big|u_t^k\big|^2\,\mathrm{d}x \leqslant 2\int \rho^k\big|u^{k-1}\big|^2\big|\nabla u^k\big|^2 + \rho^k|f|^2 + \big(\rho^k\big)^{-1}\big|Lu^k + \nabla p^k\big|^2\,\mathrm{d}x.$$

Hence, recalling from Theorem 1 that $\rho^k \in C([0,T]\times\overline{\Omega})$, $p^k \in C([0,T];H^1)$ and $u^k \in C([0,T];D_0^1\cap D^2)$, we obtain:

$$\limsup_{\tau\to 0} A_K(\tau) \leqslant C(1+\mathcal{C}_0),$$

where $\mathcal{C}_0 = \mathcal{C}(\rho_0,u_0)$ was defined previously in (3.3). Therefore, letting $\tau \to 0$ in (3.17), we conclude that

$$A_K(t) \leqslant C(1+\mathcal{C}_0) \exp\left(C \int_0^t M\big(\Phi_K(s)\big)\,\mathrm{d}s\right). \tag{3.18}$$

From (3.16) and (3.18), it follows that

$$\big|\sqrt{\rho^k}\,u^k(t)\big|_{L^2}^2 + \int_0^t \big|u_t^k(s)\big|_{D_0^1}^2\,\mathrm{d}s \leqslant C(1+\mathcal{C}_0)\exp\left[C\int_0^t M\big(\Phi_K(s)\big)\,\mathrm{d}s\right] \tag{3.19}$$

for all k, $1\leqslant k\leqslant K$.

Finally, we recall from (2.11) that

$$\big|\rho^k(t)\big|_{H^1\cap W^{1,q_0}} \leqslant C\exp\left(C\int_0^t \big|\nabla u^{k-1}(s)\big|_{H^1\cap W^{1,q_0}}\,\mathrm{d}s\right) \tag{3.20}$$

for all k, $1\leqslant k\leqslant K$. To estimate $|\nabla u^k|_{W^{1,q_0}}$ for $1\leqslant k\leqslant K$, we invoke the elliptic regularity result (2.22) and the estimate (3.12). If $3<q_0<6$, then

$$\begin{aligned}
\big|\nabla u^k\big|_{W^{1,q_0}} &\leqslant C\big|\rho^k\big(f - u_t^k - u^{k-1}\cdot\nabla u^k\big) - \nabla p^k\big|_{L^{q_0}} + C\big|\nabla u^k\big|_{L^{q_0}}\\
&\leqslant C\Big(\big|\rho^k\big|_{L^\infty}|f|_{L^{q_0}} + \big|\rho^k\big|_{L^\infty}^{(2-\theta_1)/2}\big|\sqrt{\rho^k}\,u_t^k\big|_{L^2}^{\theta_1}\big|\nabla u_t^k\big|_{L^2}^{1-\theta_1}\\
&\qquad + \big|\rho^k\big|_{L^\infty}\big|u^{k-1}\big|_{L^6}\big|\nabla u^k\big|_{L^{6q_0/(6-q_0)}} + \big|\nabla p^k\big|_{L^{q_0}}\Big) + C\big|\nabla u^k\big|_{H^1}
\end{aligned}$$

$$
\begin{aligned}
&\leqslant C\big(M(\Phi_K)A_K + |f|_{L^{q_0}}^2 + \Phi_K^{(2-\theta_1)/(1+\theta_1)} A_K^{\theta_1/(1+\theta_1)} + \big|\nabla u_t^k\big|_{L^2}^2 \\
&\qquad + \Phi_K^2 \big|\nabla u^k\big|_{L^2}^{\theta_2} \big|\nabla u^k\big|_{W^{1,q_0}}^{1-\theta_2}\big) \\
&\leqslant C\big(M(\Phi_K)A_K + |f|_{L^{q_0}}^2 + \big|\nabla u_t^k\big|_{L^2}^2\big) + \frac{1}{2}\big|\nabla u^k\big|_{W^{1,q_0}}
\end{aligned}
$$

for some $\theta_1, \theta_2 \in (0,1)$. Thus we have:

$$
\big|\nabla u^k\big|_{W^{1,q_0}} \leqslant C\big(M(\Phi_K)A_K + |f|_{L^{q_0}}^2 + \big|\nabla u_t^k\big|_{L^2}^2\big). \tag{3.21}
$$

If $q_0 = 6$, then for some $r \in (3, 6)$,

$$
\begin{aligned}
\big|\nabla u^k\big|_{W^{1,6}} &\leqslant C\big|\rho^k\big|_{L^\infty}\big(|f|_{L^6} + C\big|\nabla u_t^k\big|_{L^2}\big) + C\big|\rho^k\big|_{L^\infty}\big|\nabla u^{k-1}\big|_{L^2}\big|\nabla u^k\big|_{W^{1,r}} \\
&\quad + C\big|\nabla p^k\big|_{L^6} + C\big|\nabla u^k\big|_{L^6} \\
&\leqslant C\big(M(\Phi_K)A_K + |f|_{L^6}^2 + \big|\nabla u_t^k\big|_{L^2}^2\big) + \Phi_K^2\big|\nabla u^k\big|_{H^1}^{\theta_3}\big|\nabla u^k\big|_{W^{1,6}}^{1-\theta_3}
\end{aligned}
$$

for some $\theta_3 \in (0,1)$. Thus we also obtain:

$$
|u^k|_{W^{1,6}} \leqslant C\big(M(\Phi_K)A_K + |f|_{L^6}^2 + \big|\nabla u_t^k\big|_{L^2}^2\big). \tag{3.22}
$$

Substituting (3.12), (3.19), (3.21) and (3.22) into (3.20), we deduce that

$$
\big|\rho^k(t)\big|_{H^1\cap W^{1,q_0}} \leqslant C\exp\left[C(1+\mathcal{C}_0)\exp\left(C\int_0^t M(\Phi_K)\,\mathrm{d}s\right)\right] \tag{3.23}
$$

for all k, $1 \leqslant k \leqslant K$. Therefore, we conclude from (3.13) and (3.23) that

$$
\Phi_K(t) \leqslant C\exp\left[C(1+\mathcal{C}_0)\exp\left(C\int_0^t M(\Phi_K)\,\mathrm{d}s\right)\right]
$$

for some increasing continuous function $M = M(\cdot):[0,\infty) \to [0,\infty)$. Hence if we define $\Psi_K(t) = \log(C^{-1}\log(C^{-1}\Phi_K(t)))$, then we have:

$$
\Psi_K(t) \leqslant \log(1+\mathcal{C}_0) + C\int_0^t M\big(C\exp\big(C\exp\big(\Psi_K(s)\big)\big)\big)\,\mathrm{d}s.
$$

Thanks to this integral inequality, we can easily show that there exists a small time $T_1 \in (0, T)$ depending only on $\mathcal{C}_0$ and parameters of C such that $\Phi_K(T_1) \leqslant C\exp(C\mathcal{C}_0)$. See the proof of Lemma 6 in [18]. Moreover, from the estimates (3.12), (3.13), (3.21), (3.19) and (3.23), we derive the following uniform bound:

$$\sup_{0\leqslant t\leqslant T_1}\left(\left|\rho^k\right|_{H^1\cap W^{1,q_0}}+\left|\rho_t^k\right|_{L^2\cap L^{q_0}}+\left|u^k\right|_{D_0^1\cap D^2}+\left|\sqrt{\rho^k}\,u_t^k\right|_{L^2}\right)$$
$$+\int_0^{T_1}\left(\left|u^k\right|_{D^{2,q_0}}^2+\left|u_t^k\right|_{D_0^1}^2\right)\mathrm{d}t\leqslant C\exp\left[C\exp(C\mathcal{C}_0)\right] \tag{3.24}$$

for all $k\geqslant 1$.

3.2. *Convergence*

We show that the full sequence (ρ^k,u^k) of approximate solutions converges to a solution to the original problem (1.1), (1.2) in a strong sense. To prove this, let us define:

$$\bar\rho^{k+1}=\rho^{k+1}-\rho^k\quad\text{and}\quad\bar u^{k+1}=u^{k+1}-u^k.$$

Then it follows from the linearized momentum equation (3.5) that

$$\rho^{k+1}\bar u_t^{k+1}+\rho^{k+1}u^k\cdot\nabla\bar u^{k+1}+L\bar u^{k+1}+\nabla\left(p^{k+1}-p^k\right)$$
$$=\bar\rho^{k+1}\left(f-u_t^k-u^k\cdot\nabla u^k\right)-\rho^k\bar u^k\cdot\nabla u^k.$$

Multiplying this by $\bar u^{k+1}$, integrating over Ω and using the linearized continuity equation (3.4) and Young's inequality, we obtain:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\rho^{k+1}\left|\bar u^{k+1}\right|^2\mathrm{d}x+C^{-1}\int\left|\nabla\bar u^{k+1}\right|^2\mathrm{d}x$$
$$\leqslant C\int\left|\bar\rho^{k+1}\right|\left|f-u_t^k-u^k\cdot\nabla u^k\right|\left|\bar u^{k+1}\right|$$
$$+\rho^k\left|\bar u^k\right|\left|\nabla u^k\right|\left|\bar u^{k+1}\right|+\left|p^{k+1}-p^k\right|^2\mathrm{d}x.$$

This inequality can be derived rigorously by means of the cut-off function φ_R as in the proof of Remark 4. Then using Hölder and Sobolev inequalities together with the uniform bound (3.24), we have:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left|\sqrt{\rho^{k+1}}\,\bar u^{k+1}\right|_{L^2}^2+C^{-1}\left|\nabla\bar u^{k+1}\right|_{L^2}^2$$
$$\leqslant C\left|\bar\rho^{k+1}\right|_{L^{3/2}}^2\left|u_t^k\right|_{L^6}^2+\widetilde{C}\left(1+|f|_{L^3}^2+\left|\nabla u^k\right|_{L^2}^2\left|\nabla u^k\right|_{H^1}^2\right)\left|\bar\rho^{k+1}\right|_{L^2}^2$$
$$+C\left|\rho^k\right|_{L^\infty}\left|\sqrt{\rho^k}\,\bar u^k\right|_{L^2}^2\left|\nabla u^k\right|_{L^3}^2$$

and thus

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\big|\sqrt{\rho^{k+1}}\,\bar{u}^{k+1}\big|_{L^2}^2+C^{-1}\big|\nabla\bar{u}^{k+1}\big|_{L^2}^2\\&\quad\leqslant B^k(t)\big(\big|\bar{\rho}^{k+1}\big|_{L^{3/2}}^2+\big|\bar{\rho}^{k+1}\big|_{L^2}^2\big)+\widetilde{C}\big|\sqrt{\rho^k}\,\bar{u}^k\big|_{L^2}^2,\end{aligned}\tag{3.25}$$

where $B^k(t)=\widetilde{C}(1+|f|_{L^3}^2+|\nabla u_t^k|_{L^2}^2)$. Note that $\int_0^{T_1}B^k(t)\,\mathrm{d}t\leqslant\widetilde{C}$ for all $k\geqslant 1$, thanks to the uniform bound (3.24). Here we denote by $\widetilde{C}$ a generic positive constant depending only on $\mathcal{C}_0$ and parameters of C.

On the other hand, we observe from (3.4) that

$$\bar{\rho}_t^{k+1}+\operatorname{div}\big(\bar{\rho}^{k+1}u^k\big)+\operatorname{div}\big(\rho^k\bar{u}^k\big)=0.\tag{3.26}$$

Then using the cut-off function φ_R, we easily prove that $\bar{\rho}^{k+1}\in L^\infty(0,T_1;L^{3/2})$. Hence multiplying (3.26) by $\operatorname{sgn}(\bar{\rho}^{k+1})|\bar{\rho}^{k+1}|^{1/2}$ and integrating over Ω, we get:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int\big|\bar{\rho}^{k+1}\big|^{3/2}\,\mathrm{d}x&\leqslant C\int\big|\nabla u^k\big|\big|\bar{\rho}^{k+1}\big|^{3/2}+\big(\big|\nabla\rho^k\big|\big|\bar{u}^k\big|+\rho^k\big|\nabla\bar{u}^k\big|\big)\big|\bar{\rho}^{k+1}\big|^{1/2}\,\mathrm{d}x\\&\leqslant C\big|\nabla u^k\big|_{L^\infty}\big|\bar{\rho}^{k+1}\big|_{L^{3/2}}^{3/2}+C\big|\rho^k\big|_{H^1}\big|\nabla\bar{u}^k\big|_{L^2}\big|\bar{\rho}^{k+1}\big|_{L^{3/2}}^{1/2}.\end{aligned}$$

Multiplying $|\bar{\rho}^{k+1}|_{L^{3/2}}^{1/2}$ on both sides, we have:

$$\frac{\mathrm{d}}{\mathrm{d}t}\big|\bar{\rho}^{k+1}\big|_{L^{3/2}}^2\leqslant D_\varepsilon^k(t)\big|\bar{\rho}^{k+1}\big|_{L^{3/2}}^2+\varepsilon\big|\nabla\bar{u}^k\big|_{L^2}^2,\tag{3.27}$$

where $D_\varepsilon^k(t)=C_\varepsilon|\rho^k(t)|_{H^1}^2+C|\nabla u^k(t)|_{L^\infty}$. It follows from (3.24) that $\int_0^t D_\varepsilon^k(t)\,\mathrm{d}t\leqslant\widetilde{C}+\widetilde{C}_\varepsilon t$ for all $t\leqslant T_1$ and $k\geqslant 1$. Similarly, we deduce that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\big|\bar{\rho}^{k+1}\big|_{L^2}^2&\leqslant C\big|\bar{\rho}^{k+1}\big|_{L^2}^2\big|\nabla u^k\big|_{L^\infty}+C\big(\big|\nabla\rho^k\big|_{L^3}+\big|\rho^k\big|_{L^\infty}\big)\big|\bar{\rho}^{k+1}\big|_{L^2}\big|\nabla\bar{u}^k\big|_{L^2}\\&\leqslant E_\varepsilon^k(t)\big|\bar{\rho}^{k+1}\big|_{L^2}^2+\varepsilon\big|\nabla\bar{u}^k\big|_{L^2},\end{aligned}\tag{3.28}$$

where $E_\varepsilon^k(t)=C|\nabla u^k(t)|_{L^\infty}+C_\varepsilon(|\rho^k(t)|_{L^\infty}+|\nabla\rho^k(t)|_{L^3})^2$. It follows also from (3.24) that $\int_0^t E_\varepsilon^k(t)\,\mathrm{d}t\leqslant\widetilde{C}+\widetilde{C}_\varepsilon t$ for all $t\leqslant T_1$ and $k\geqslant 1$.

Combining (3.25), (3.27) and (3.28), we obtain:

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\big(\big|\sqrt{\rho^{k+1}}\,\bar{u}^{k+1}\big|_{L^2}^2+\big|\bar{\rho}^{k+1}\big|_{L^{3/2}}^2+\big|\bar{\rho}^{k+1}\big|_{L^2}^2\big)+C^{-1}\big|\nabla\bar{u}^{k+1}\big|_{L^2}^2\\&\quad\leqslant C\big|\sqrt{\rho^k}\,\bar{u}^k\big|_{L^2}^2+F_\varepsilon^k(t)\big(|\bar{\rho}^{k+1}\big|_{L^{3/2}}^2+\big|\bar{\rho}^{k+1}\big|_{L^2}^2\big)+\varepsilon\big|\nabla\bar{u}^k\big|_{L^2}^2\end{aligned}\tag{3.29}$$

for some function $F_\varepsilon^k(t)$ with $\int_0^t F_\varepsilon^k(t)\,\mathrm{d}t\leqslant\widetilde{C}+\widetilde{C}_\varepsilon t$ for all $t\leqslant T_1$ and $k\geqslant 1$. Let us define:

$$\varphi^{k+1}(t)=C\big(\big|\sqrt{\rho^{k+1}}\,\bar{u}^{k+1}(t)\big|_{L^2}^2+\big|\bar{\rho}^{k+1}(t)\big|_{L^{3/2}}^2+\big|\bar{\rho}^{k+1}(t)\big|_{L^2}^2\big)$$

and $\psi^{k+1}(t) = |\nabla \bar{u}^{k+1}(t)|_{L^2}^2$. Then integrating (3.29) over $(0,t) \subset (0,T_1)$, we have:

$$\varphi^{k+1}(t) + \int_0^t \psi^{k+1}\,\mathrm{d}s \leqslant C \int_0^t \bigl(\varphi^k + \varepsilon\psi^k\bigr)\,\mathrm{d}s + \int_0^t F_\varepsilon^k \varphi^{k+1}\,\mathrm{d}s,$$

which implies, by virtue of Gronwall's inequality, that

$$\varphi^{k+1}(t) + \int_0^t \psi^{k+1}\,\mathrm{d}s \leqslant \widetilde{C}\exp\bigl(\widetilde{C}_\varepsilon t\bigr) \int_0^t \bigl(\varphi^k + \varepsilon\psi^k\bigr)\,\mathrm{d}s.$$

Hence, choosing $\varepsilon > 0$ and then $T_* > 0$ so small that $4(T_* + \varepsilon)\widetilde{C} < 1$, $T_* < T_1$ and $\exp(\widetilde{C}_\varepsilon T_*) < 2$, we also deduce from Gronwall's inequality that for all $K \geqslant 1$,

$$\sum_{k=1}^K \left(\sup_{0\leqslant t\leqslant T_*} \varphi^{k+1}(t) + \int_0^{T_*} \psi^{k+1}(t)\,\mathrm{d}t \right) < \infty.$$

Therefore, recalling from (2.8) that $\rho^{k+1} \geqslant \delta C^{-1}$, we conclude that (ρ^k, u^k) converges to a limit (ρ, u) in the following strong sense:

$$u^k \to u \quad \text{in } L^\infty\bigl(0,T_*;L^2\bigr) \cap L^2\bigl(0,T_*;D_0^1\bigr) \quad \text{and} \quad \rho^k \to \rho \quad \text{in } L^\infty\bigl(0,T_*;L^2\bigr).$$

Now it is a simple matter to check that (ρ, u) is a weak solution to the original problem (1.1), (1.2). Then, by virtue of the lower semi-continuity of norms, we deduce from the uniform bound (3.24) that (ρ, u) satisfies the following regularity estimate:

$$\begin{aligned} &\operatorname*{ess\,sup}_{0\leqslant t\leqslant T_*}\bigl(|\rho|_{H^1\cap W^{1,q_0}} + |\rho_t|_{L^2\cap L^{q_0}} + |u|_{D_0^1\cap D^2} + \bigl|\sqrt{\rho}\,u_t\bigr|_{L^2}\bigr) \\ &\quad + \int_0^{T_*} \bigl(|u|_{D^{2,q_0}}^2 + |u_t|_{D_0^1}^2\bigr)\,\mathrm{d}t \leqslant C\exp\bigl[C\exp(C\mathcal{C}_0)\bigr]. \end{aligned} \tag{3.30}$$

The time-continuity of the solution (ρ, u) can be proved by the same argument as in Section 2.3. This completes the proof of Proposition 5.

Remark 6. It should be emphasized that the constant C and the local existence time T_* in (3.30) don't depend on δ and the size of Ω.

4. Statements and proof of main results

In this section, we state and prove all of our main results. We first prove an existence result and a blow-up criterion on local strong solutions for general nonnegative initial densities with minimal regularity.

Theorem 7. *Assume that $p = p(\cdot) \in C^1[0,\infty)$, and the data (ρ_0, u_0, f) satisfy the regularity conditions*:

$$\begin{aligned}
&\rho_0 \in H^1 \cap W^{1,q}, \quad u_0 \in D_0^1 \cap D^2, \\
&f \in C\big([0,T]; L^2\big) \cap L^2\big(0,T; L^q\big) \quad \textit{and} \quad f_t \in L^2\big(0,T; H^{-1}\big)
\end{aligned} \tag{4.1}$$

for some q with $3 < q < \infty$ and the compatibility condition:

$$Lu_0 + \nabla p(\rho_0) = \rho_0^{1/2} g \quad \textit{for some } g \in L^2. \tag{4.2}$$

Then there exist a time $T_* \in (0,T)$ and a unique strong solution (ρ,u) to the initial boundary value problem (1.1), (1.2) *such that*

$$\begin{aligned}
&\rho \in C\big([0,T_*]; H^1 \cap W^{1,q_0}\big), \quad \rho_t \in C\big([0,T_*]; L^2 \cap L^{q_0}\big), \\
&u \in C\big([0,T_*]; D_0^1 \cap D^2\big) \cap L^2\big(0,T_*; D^{2,q_0}\big), \\
&u_t \in L^2\big(0,T_*; D_0^1\big) \quad \textit{and} \quad \sqrt{\rho}\, u_t \in L^\infty\big(0,T_*; L^2\big),
\end{aligned} \tag{4.3}$$

where $q_0 = \min(6,q)$. Furthermore, we have the following blow-up criterion: *If T^* is the maximal existence time of the strong solution (ρ,u) and $T^* < T$, then*

$$\limsup_{t \nearrow T^*} \big(|\rho(t)|_{H^1 \cap W^{1,q_0}} + |u(t)|_{D_0^1}\big) = \infty. \tag{4.4}$$

Proof. Assume for the moment that Ω is a bounded domain with smooth boundary. Let (ρ_0, u_0, f) be the given data satisfying (4.1) and (4.2). For each small $\delta > 0$, let $\rho_0^\delta = \rho_0 + \delta$ and let $u_0^\delta \in H_0^1 \cap H^2$ be the unique solution to the elliptic problem:

$$Lu_0^\delta = -\nabla p\big(\rho_0^\delta\big) + \big(\rho_0^\delta\big)^{1/2} g \quad \text{in } \Omega.$$

Then by virtue of Proposition 5, there exist a time $T_* \in (0,T)$ and a unique strong solution (ρ^δ, u^δ) in $[0,T_*] \times \Omega$ to the problem (1.1), (1.2) with the initial data replaced by $(\rho_0^\delta, u_0^\delta)$. Note that $u_0^\delta \to u_0$ in H^2 as $\delta \to 0$, (ρ^δ, u^δ) satisfies the bound (3.2) with $\mathcal{C}_0 = |g|_{L^2}^2$, and the constants T_*, C and $\mathcal{C}_0$ are independent of δ. Hence, following the same arguments as in the proof of Theorem 1 (see Sections 2.1 and 2.3), we prove the existence and regularity of a strong solution to the original problem (1.1), (1.2). Moreover, since the constants C, $\mathcal{C}_0$ and the local existence time T_* in (3.2) are independent of the size of the domain, we also obtain the same existence and regularity results for unbounded domains by means of the domain expansion technique (see Section 2.2). Finally, the uniqueness can be proved by using the similar (even easier) methods to the proof of the convergence in Section 3.2. This proves the first part of Theorem 7 and it remains to prove the blow-up criterion (4.4).

To prove this, suppose that $T^* < T$, and let us introduce functions, defined by:

$$\Phi(t) = 1 + \left|\rho(t)\right|_{H^1\cap W^{1,q_0}} + \left|u(t)\right|_{D_0^1}$$

and

$$J(t) = 1 + \left|\rho(t)\right|_{H^1\cap W^{1,q_0}} + \left|\rho_t(t)\right|_{L^2\cap L^{q_0}} + \left|u(t)\right|_{D_0^1\cap D^2} + \left|\sqrt{\rho}\,u_t(t)\right|_{L^2} + \int_0^t \left(\left|u(s)\right|^2_{D^{2,q_0}} + \left|u_t(s)\right|^2_{D_0^1}\right) \mathrm{d}s$$

for $0 < t < T^*$. Let τ be a fixed time in $(0, T^*)$. Then (ρ, u) is a strong solution of Eqs. (1.1) in $[\tau, T^*) \times \Omega$, which satisfies the regularity (4.3). Hence, following exactly the same arguments as in Section 3, we can prove the analogues of (3.12), (3.16), (3.20) and (3.21): for each $t \in (\tau, T^*)$,

$$\left|\nabla u(t)\right|_{H^1} \leqslant C\left(1 + \left|\sqrt{\rho}\,u_t(t)\right|_{L^2}\right) M\left(\Phi(t)\right), \tag{4.5}$$

$$\left|\sqrt{\rho}\,u_t(t)\right|^2_{L^2} + \int_\tau^t \left|\nabla u_t\right|^2_{L^2} \mathrm{d}s \leqslant C + C\left|\sqrt{\rho}\,u_t(\tau)\right|^2_{L^2} + C\int_\tau^t \left(1 + \left|\sqrt{\rho}\,u_t(t)\right|^2_{L^2}\right) M(\Phi)\, \mathrm{d}s, \tag{4.6}$$

$$\left|\rho(t)\right|_{H^1\cap W^{1,q_0}} \leqslant C \exp\left(C \int_0^t \left|\nabla u(s)\right|_{H^1\cap W^{1,q_0}} \mathrm{d}s\right) \tag{4.7}$$

and

$$\left|\nabla u(t)\right|_{W^{1,q_0}} \leqslant C\left(\left(1 + \left|\sqrt{\rho}\,u_t(t)\right|^2_{L^2}\right) M\left(\Phi(t)\right) + \left|f(t)\right|^2_{L^{q_0}} + \left|\nabla u_t(t)\right|^2_{L^2}\right) \tag{4.8}$$

for an increasing continuous function $M : [0, \infty) \to [0, \infty)$.

In view of Gronwall's inequality, we deduce from (4.6) that

$$\left|\sqrt{\rho}\,u_t(t)\right|^2_{L^2} + \int_0^t \left|\nabla u_t\right|^2_{L^2} \mathrm{d}s \leqslant C J(\tau) \exp\left(C T^* \sup_{0\leqslant s\leqslant t} M\left(\Phi(s)\right)\right). \tag{4.9}$$

Combining (4.5), (4.7)–(4.9), and using the continuity equation, we conclude that for each $t \in (\tau, T^*)$,

$$J(t) \leqslant CJ(\tau)\Big(\sup_{0\leqslant s\leqslant t} M\big(\Phi(s)\big)\Big)\exp\Big(CT^* \sup_{0\leqslant s\leqslant t} M\big(\Phi(s)\big)\Big). \tag{4.10}$$

Now the blow-up criterion (4.4) follows immediately from (4.10) because the maximality of T^* implies that $J(t) \to \infty$ as $t \to T_*$. This completes the proof of Theorem 7. □

Adapting the proof of Theorem 7, we can also prove:

Theorem 8. *Assume that $p = p(\cdot) \in C^2[0,\infty)$ and the data (ρ_0, u_0, f) satisfy the hypotheses of Theorem 7. If in addition, $\rho_0 \in H^2$ and $f_t \in L^2(0,T;L^2)$, then there exist a time $T_* > 0$ and a unique strong solution (ρ, u) satisfying the regularity*:

$$\rho \in C\big([0,T_*];H^2\big), \quad \rho_t \in C\big([0,T_*];H^1\big) \quad \textit{and} \quad u \in L^2\big(0,T_*;D^3\big)$$

as well as (4.3). Furthermore, we have the following blow-up criterion: If T^* is the maximal existence time of the strong solution (ρ, u) and $T^* < T$, then

$$\limsup_{t\nearrow T^*}\big(|\rho(t)|_{H^2} + |u(t)|_{D_0^1}\big) = \infty. \tag{4.11}$$

Proof. The proof follows basically the same arguments as in the proof of Theorem 7. Hence we only give a very brief indication of how to modify it.

The key estimates for the proof of Theorem 7 are (3.12), (3.16), (3.20) and (3.21). To prove Theorem 8, we define a new functional $\widetilde{\Phi}_K$ by:

$$\widetilde{\Phi}_K(t) = \max_{1\leqslant k\leqslant K}\sup_{0\leqslant s\leqslant t}\big(1 + |\rho^k(s)|_{H^2} + |u^k(s)|_{D_0^1}\big).$$

Then it follows immediately from (3.12) and (3.16) that

$$|\nabla u^k|_{H^1} \leqslant CM\big(\widetilde{\Phi}_K\big)\big(1 + |\sqrt{\rho^k}\,u_t^k|_{L^2}\big) \tag{4.12}$$

and

$$|\sqrt{\rho^k}\,u_t^k(t)|_{L^2}^2 + \int_\tau^t |\nabla u_t^k|^2\,\mathrm{d}s \leqslant C\left(A_K(\tau) + \int_\tau^t M\big(\widetilde{\Phi}_K\big)A_K\,\mathrm{d}s\right). \tag{4.13}$$

But the estimates (3.20) and (3.21) should be replaced by slightly stronger ones. First, if we differentiate the linearized continuity equation (3.4) with respect to x_i and x_j, multiply by $\partial_i\partial_j\rho^k$ and then integrate over Ω, we can obtain:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\left|\partial_i\partial_j\rho^k\right|^2\mathrm{d}x \leqslant C\int\left|\nabla u^{k-1}\right|\left|\nabla^2\rho^k\right|^2+\left|\nabla^2 u^{k-1}\right|\left|\nabla\rho^k\right|\left|\nabla^2\rho^k\right| + \rho^k\left|\nabla^2\operatorname{div}u^{k-1}\right|\left|\nabla^2\rho^k\right|\mathrm{d}x$$
$$\leqslant C\left|\nabla u^{k-1}\right|_{H^2}\left|\rho^k\right|_{H^2}^2.$$

Treating the lower order terms ρ^k and $\partial_i\rho^k$ in a similar way, we have:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left|\rho^k\right|_{H^2}^2 \leqslant C\left|\nabla u^{k-1}\right|_{H^2}\left|\rho^k\right|_{H^2}^2.$$

Hence using (3.20) and Gronwall's inequality, we derive:

$$\left|\rho^k(t)\right|_{H^2} \leqslant \widetilde{C}\exp\left(\int_0^t\left|\nabla u^{k-1}(s)\right|_{H^2}\mathrm{d}s\right), \tag{4.14}$$

where $\widetilde{C}$ is a positive constant depending only on $|\rho_0|_{H^2}$ and parameters of C. To derive a stronger estimate than (3.21), we use the following elliptic regularity estimate (see Lemma 14 below):

$$\left|u^k\right|_{D^2\cap D^3} \leqslant C\left|\rho^k f-\rho^k u_t^k-\rho^k u^{k-1}\cdot\nabla u^k-\nabla p^k\right|_{H^1}+C\left|u^k\right|_{D_0^1}. \tag{4.15}$$

Then, since $|\nabla p^k|_{H^1}\leqslant CM(\widetilde{\Phi}_K)$, we can deduce from (4.12) and (4.15) that for any $1\leqslant k\leqslant K$,

$$|\nabla u_k|_{H^2} \leqslant C\left(M\left(\widetilde{\Phi}_K\right)A_K+|f|_{H^1}^2+\left|\nabla u_t^k\right|_{L^2}^2\right). \tag{4.16}$$

Based on the key estimates (4.12)–(4.14) and (4.16), we can prove Theorem 8 by following exactly the same arguments as in the proof of Theorem 7. □

Our final result is concerned with the initial condition (1.2) and the compatibility condition (4.2). To state the result precisely, we denote by V the *initial vacuum*, i.e., the interior of the zero-set of the initial density in Ω.

Theorem 9. *Let* (ρ_0, u_0, f) *be the given data satisfying the regularity condition* (4.1), *and assume that either* V *is empty or the elliptic system,*

$$Lw=-\mu\Delta w-(\lambda+\mu)\nabla\operatorname{div}w=0, \tag{4.17}$$

has only one solution w *in* $D_0^1(V)\cap D^2(V)$. *Then there exists a unique* (*local*) *strong solution* (ρ, u) *with the regularity* (4.3) *such that*

$$\left|\rho(t)-\rho_0\right|_{H^1\cap W^{1,q_0}}+\left|u(t)-u_0\right|_{D_0^1\cap D^2}\to 0 \quad \text{as } t\to 0, \tag{4.18}$$

if and only if the initial data satisfy the compatibility condition (4.2).

It should be noted that the solutions of the system (4.17) in V are unique provided that V has a sufficiently simple geometry, for instance, it consists of a finite number of domains with Lipschitz boundary.

Proof. Let (ρ, u) be a strong solution to the problem (1.1), (1.2) with the regularity (4.3). Then, since $\sqrt{\rho}\, u_t \in L^\infty(0, T_*; L^2)$, we can find a sequence $\{t_k\}$, $t_k \to 0$, such that the sequence $(\sqrt{\rho}\, u_t(t_k))$ converges weakly in L^2. Therefore, letting $t_k \to 0$ in the momentum equation, we can deduce that

$$-\mu\Delta u(0) - (\lambda + \mu)\nabla \operatorname{div} u(0) + \nabla p\big(\rho(0)\big) = \rho(0)^{1/2}\tilde{g} \tag{4.19}$$

for some $\tilde{g} \in L^2$. This proves the necessity of the condition (4.2) thanks to the convergence (4.18).

To prove the sufficiency, let (ρ_0, u_0) be the initial data satisfying the conditions (4.1) and (4.2). Then there exists a unique strong solution (ρ, u) satisfying $\rho \in C([0, T_*]; H^1 \cap W^{1,q_0})$ and $u \in C([0, T_*]; D_0^1 \cap D^2)$. Hence we have only to show that $\rho(0) = \rho_0$ and $u(0) = u_0$ in Ω. On the other hand, it can be easily deduced from the weak formulation that $\rho(0) = \rho_0$ and $(\rho u)(0) = \rho_0 u_0$ in Ω and it remains to show that $u(0) = u_0$ in the initial vacuum V. Define $w = u(0) - u_0$. Then since $(\rho(0), u(0))$ also satisfies the relation (4.19) for some $\tilde{g} \in L^2$, we find that $w \in D_0^1(V) \cap D^2(V)$ is a solution to the elliptic problem (4.17) in V and thus $w \equiv 0$ in V. This completes the proof of the theorem. □

Remark 10. Adapting the arguments in this paper, we can prove similar results for bounded domains in $\mathbb{R}^2$.

5. Some regularity results on the Lamé system

In this final section, we derive some regularity estimates for the so-called Lamé system:

$$Lu = -\mu\Delta u - (\lambda + \mu)\nabla \operatorname{div} u = F \quad \text{in } \Omega, \tag{5.1}$$

where Ω is a bounded or unbounded domain in $\mathbb{R}^3$. All of our results rely crucially on these estimates.

First, from a well-known elliptic theory due to S. Agmon, A. Douglis, and L. Nirenberg in [1], we recall the following result.

Lemma 11. *Assume that Ω is a bounded domain in $\mathbb{R}^3$ with smooth boundary, and let $u \in W_0^{1,q}(\Omega)$ be a weak solution of the system* (5.1)*, where $1 < q < \infty$. If $F \in W^{k,q}(\Omega)$ for $k \geqslant 0$, then $u \in W^{k+2,q}(\Omega)$ and*

$$|u|_{W^{k+2,q}(\Omega)} \leqslant C|F|_{W^{k,q}(\Omega)} \tag{5.2}$$

for some constant $C = C(q, \mu, \lambda, \Omega)$ *independent of* F.

The main purpose of this section is to derive regularity estimates when the domain is the intersection of an unbounded domain and an open ball with a large radius.

Lemma 12. *Let* $\Omega_R = B_R$ *or* $B_R \cap \mathbb{R}^3_+$ *with* $R \geqslant 1$, *where* $B_R = \{x \in \mathbb{R}^3\colon |x| < R\}$. *If* $u \in W_0^{1,q}(\Omega_R)$, $1 < q < \infty$, *is a weak solution of the system* (5.1), *then*

$$|u|_{D^{k+2,q}(\Omega_R)} \leqslant C|F|_{W^{k,q}(\Omega_R)} \tag{5.3}$$

for some constant C *independent of* R.

Proof. If we define $v \in W_0^{1,q}(\Omega_1)$ by $v(x) = u(Rx)$ for $x \in \Omega_1$, then we have:

$$Lv(x) = R^2 Lu(Rx) = R^2 F(Rx) \equiv G(x) \quad \text{for } x \in \Omega_1.$$

Hence it follows from Lemma 11 that

$$|v|_{D^{k+2,q}(\Omega_1)} \leqslant C|G|_{W^{k,q}(\Omega_1)}.$$

Converting this back into the unscaled variables, we easily obtain (5.3). □

Lemma 13. *Let* $\Omega_R = \Omega \cap B_R$, *where* Ω *is an exterior domain in* $\mathbb{R}^3$ *with smooth boundary. Choose a fixed number* $R_0 = R_0(\Omega) \geqslant 1$ *such that* $\Omega^c \subset B_{R_0}$ *and assume* $R > 2R_0$. *If* $u \in W_0^{1,q}(\Omega_R)$, $1 < q < \infty$, *is a weak solution of the system* (5.1), *then*

$$|u|_{D^{k+2,q}(\Omega_R)} \leqslant C\big(|F|_{W^{k,q}(\Omega_R)} + |u|_{D_0^{1,q}(\Omega_R)}\big) \tag{5.4}$$

for some constant C *independent of* R.

Proof. Choosing a cut-off function $\varphi \in C_c^\infty(\mathbb{R}^3)$ such that $\varphi = 1$ in B_{R_0} and $\varphi = 0$ on $B^c_{2R_0}$, we define:

$$v = \varphi u \quad \text{and} \quad w = (1-\varphi)u \equiv \psi u.$$

First, observing that $v \in W_0^{1,q}(\Omega_{2R_0})$, we deduce from Lemma 11 that

$$\begin{aligned} |v|_{W^{2,q}(\Omega_{2R_0})} &\leqslant C|Lv|_{L^q(\Omega_{2R_0})} \\ &\leqslant C|\varphi Lu|_{L^q(\Omega_{2R_0})} + C|L(\varphi u) - \varphi Lu|_{L^q(\Omega_{2R_0})}. \end{aligned} \tag{5.5}$$

We easily estimate the first term in (5.5):

$$|\varphi Lu|_{L^q(\Omega_{2R_0})} \leqslant |\varphi|_{L^\infty(\mathbb{R}^3)}|Lu|_{L^q(\Omega_{2R_0})} \leqslant C|Lu|_{L^q(\Omega_R)}.$$

Using Poincaré inequality,

$$|u|_{L^q(\Omega_{2R_0})} \leqslant C|\nabla u|_{L^q(\Omega_{2R_0})}, \tag{5.6}$$

we also have:

$$\left|L(\varphi u) - \varphi Lu\right|_{L^q(\Omega_{2R_0})} \leqslant C|\nabla\varphi|_{W^{1,\infty}(\mathbb{R}^3)}|u|_{W^{1,q}(\Omega_{2R_0})} \leqslant C|\nabla u|_{L^q(\Omega_R)}.$$

Hence, substituting these estimates into (5.5), we conclude that

$$|v|_{W^{2,q}(\Omega_R)} \leqslant C\big(|Lu|_{L^q(\Omega_R)} + |\nabla u|_{L^q(\Omega_R)}\big). \tag{5.7}$$

Next, to estimate w, we observe that $w = 0$ in B_{R_0} and $w \in W_0^{1,q}(B_R)$. Then it follows from Lemma 12 and Poincaré inequality (5.6) that

$$\begin{aligned} |w|_{D^{2,q}(\Omega_B)} &\leqslant C|Lw|_{L^q(B_R)} \\ &\leqslant C|\psi Lu|_{L^q(\Omega_R)} + C\left|L(\psi u) - \psi Lu\right|_{L^q(\Omega_R)} \\ &\leqslant C|Lu|_{L^q(\Omega_R)} + C\left|L(\psi u) - \psi Lu\right|_{L^q(\Omega_{2R_0})} \\ &\leqslant C\big(|Lu|_{L^q(\Omega_R)} + |\nabla u|_{L^q(\Omega_R)}\big). \end{aligned} \tag{5.8}$$

Combining (5.7) and (5.8), we prove (5.4) with $k = 0$. Using standard calculus inequalities and adapting the above argument, we can also prove the higher regularity estimates for $k \geqslant 1$. This completes the proof of the lemma. □

Then using the domain expansion technique as in Section 2.2, we easily prove

Lemma 14. *Let Ω be the whole pace $\mathbb{R}^3$, the half space $\mathbb{R}^3_+$ or an exterior domain in $\mathbb{R}^3$ with smooth boundary. If $u \in D_0^1(\Omega)$ is a weak solution of the system* (5.1)*, then*

$$|u|_{D^{k+2,q}(\Omega)} \leqslant C\big(|F|_{W^{k,q}(\Omega)} + |u|_{D_0^{1,q}(\Omega)}\big)$$

for any q with $1 < q < \infty$.

Finally we provide a refined result for the whole space.

Lemma 15. *If $u \in D_0^{1,q}(\mathbb{R}^3)$, $1 < q < \infty$, is a weak solution of the system* (5.1)*, then*

$$|u|_{D^{k+2,q}(\mathbb{R}^3)} \leqslant C|F|_{D^{k,q}(\mathbb{R}^3)}. \tag{5.9}$$

Proof. We may assume that $u \in C_c^\infty(\mathbb{R}^3)$. Then taking the divergence operator to (5.1), we have:

$$-(\lambda + 2\mu)\Delta \operatorname{div} u = \operatorname{div} F \quad \text{and} \quad \operatorname{div} u = -(\lambda + 2\mu)^{-1}\Delta^{-1} \operatorname{div} F.$$

Hence, substituting this into (5.1), we derive a solution formula:

$$u = -\mu^{-1}\Delta^{-1}F + \mu^{-1}(\lambda + \mu)(\lambda + 2\mu)^{-1}\Delta^{-1}\nabla\Delta^{-1} \operatorname{div} F.$$

From this formula, we easily derive (5.9) by applying the classical estimates from Harmonic analysis (see Chapter 5 in [20], for instance). □

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